Structures and Earthquakes

Class Handouts

Fawad Ahmed Najam
National University of Sciences and Technology (NUST), Islamabad, Pakistan

October, 2020

Contents

Part 1:

Structural Dynamics – The Basics

Part 1: Structural Dynamics – The Basics

Table of Contents

Chapter 1

Introduction to Structural Dynamics

The basic goal of structural dynamics is to determine the structural response against a dynamic loading (excitation). The term response is used in a general sense to include any response quantity, such as displacement, velocity, or acceleration of the mass; also, an internal force or internal stress in the structure.

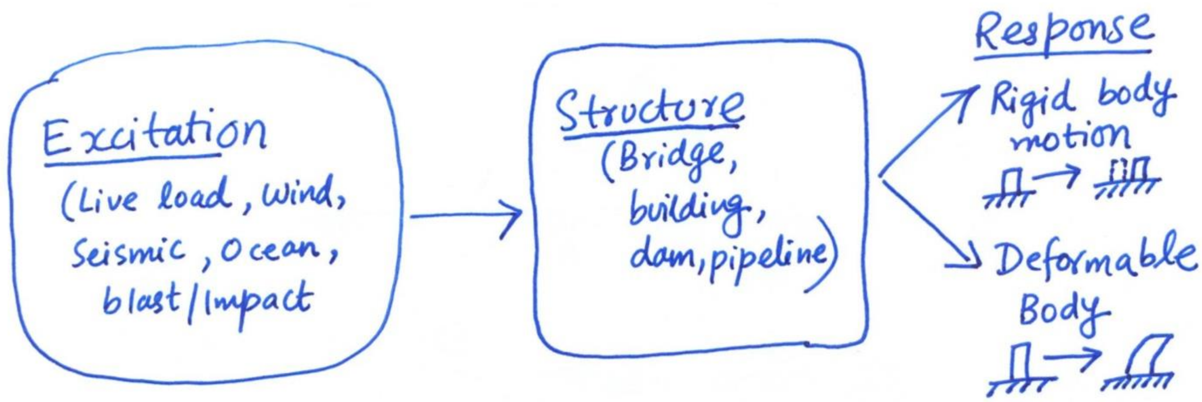


Figure 1-1: The fundamental objective of structural dynamics

1.1. Classification of Dynamic Loading

The term "dynamic" may be defined simply as time-varying; thus a dynamic load is any load of which its magnitude, direction, and/or position varies with time. Similarly, the structural response to a dynamic load, i.e., the resulting stresses and deflections, is also time-varying, or dynamic.

Two basically different approaches are available for evaluating structural response to dynamic loads: "deterministic" and "nondeterministic". The choice of method to be used in any given case depends upon how the loading is defined. If the time variation of loading is fully known, even though it may be highly oscillatory or irregular in character, it will be referred to herein as a "prescribed dynamic loading"; and the analysis of the response of any specified structural system to a prescribed dynamic loading is defined as a "deterministic analysis". On the other hand, if the time variation is not completely known but can be defined in a statistical sense, the loading is termed a "random dynamic loading"; and its corresponding analysis of response is defined as a "nondeterministic or probabilistic analysis".

In general, structural response to any dynamic loading is expressed basically in terms of the displacements of the structure. Thus, a deterministic analysis leads directly to displacement time-histories corresponding to the prescribed loading history; other related response quantities, such as stresses, strains, internal forces, etc., are usually obtained as a secondary phase of the analysis. On the other hand, a nondeterministic analysis provides only statistical information about the displacements resulting from the statistically defined loading; corresponding information on the related response quantities are then generated using independent nondeterministic analysis procedures (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

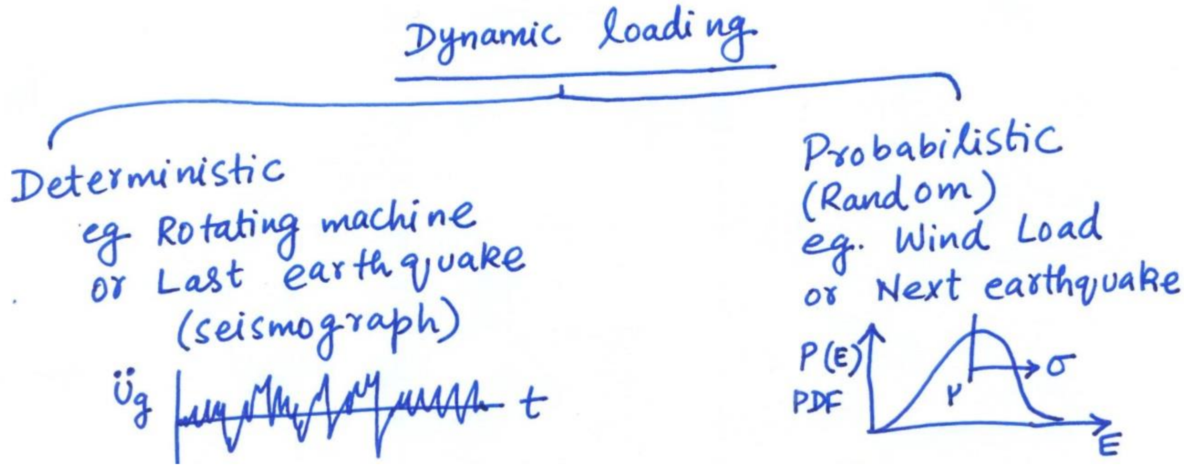


Figure 1-2: The deterministic vs. probabilistic load

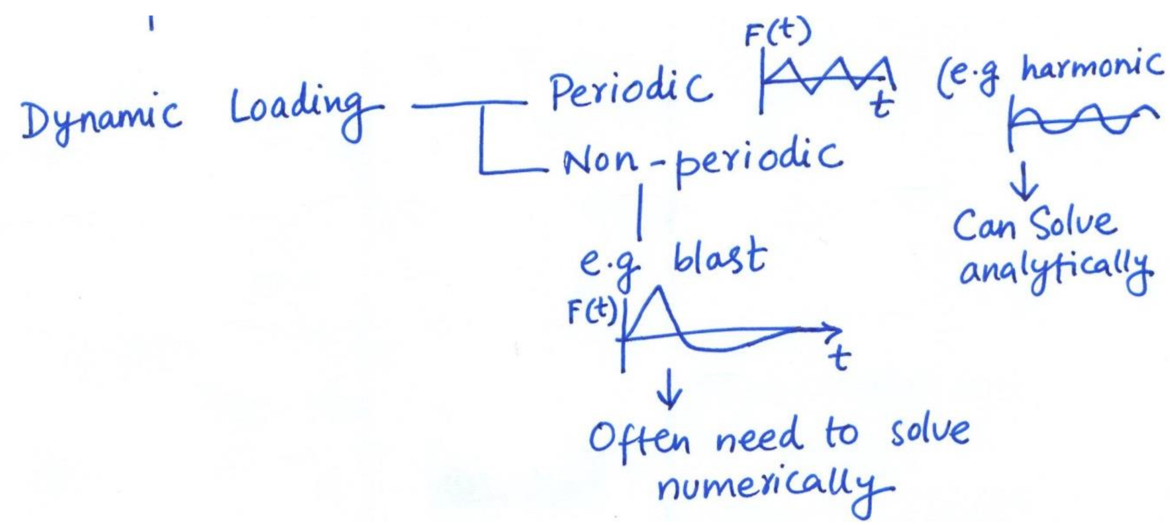


Figure 1-3: The periodic vs. non-periodic load

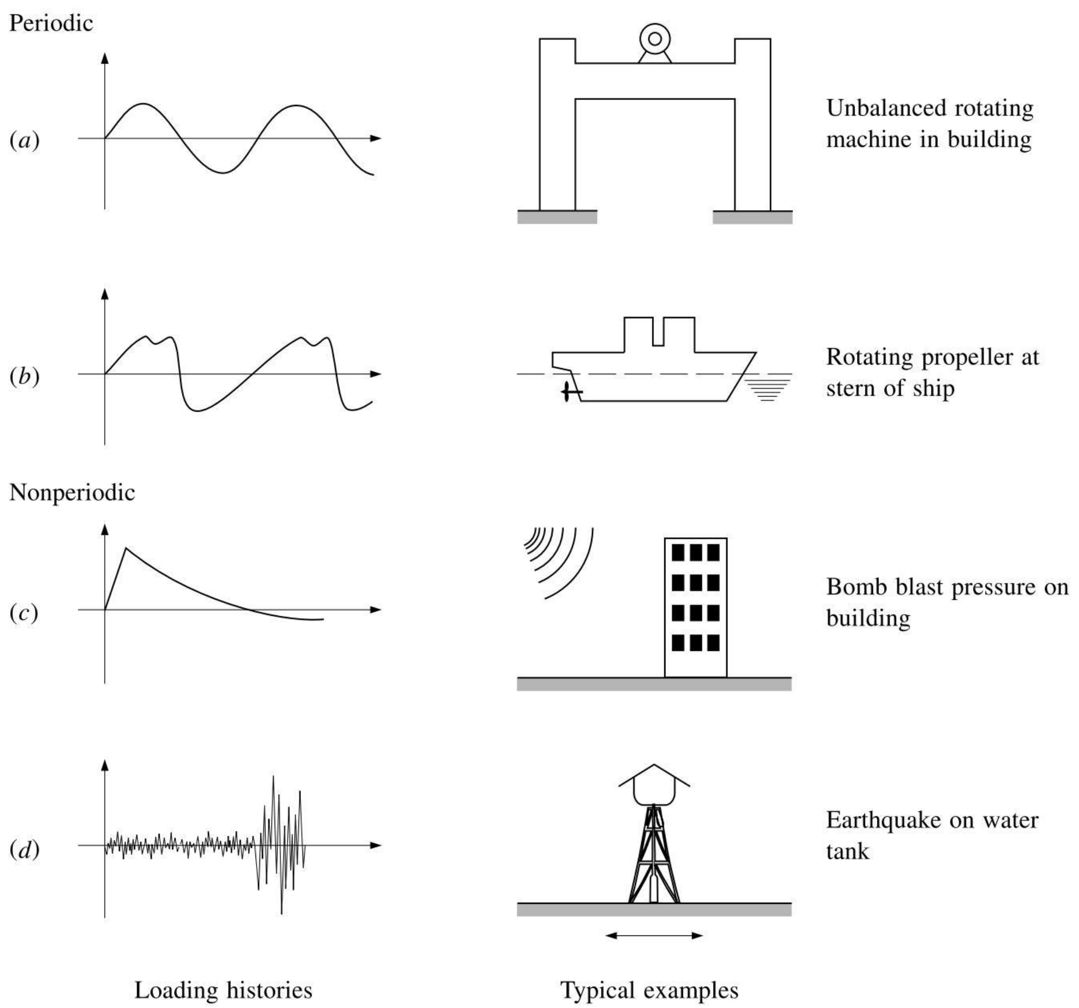


Figure 1-4: Characteristics and sources of typical dynamic loadings: (a) simple harmonic; (b) complex; (c) impulsive; (d) long-duration. (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

1.2. The Fundamental Relationships in Static Structural Analysis

There are six basic concepts that lie at the foundation of theories governing the behavior of structures, from analysis to design.

- Loads and Load Effects
- Actions
- Deformations
- Strains
- Stresses
- Stress-Resultants

Loads are the actual physical excitations that may act on the structure e.g. gravity, wind pressure, dynamic inertial effects and retention of liquid., Loads and its effects can lead to actions, (which are basically the idealized forces acting on the members) e.g. bending moment, shear force etc. Actions can lead to deformations, which again are idealized into various components such as rotation, shortening, and shearing angle Deformations cause strains which are basically normalized deformation at the cross-section material or fiber level. Strains may lead to stresses in material fibers, which generally have a correspondence with the strain through material stress-strain model. The stresses can be summed up in any particular manner to determine the internal stress resultants.

In general, for a structure to be in static or dynamic equilibrium, the internal stress resultants should be in equilibrium with the actions due to loads. An alternative way of looking at the same linkage is that the actions cause stresses in the member cross-sections. These stresses cause strains, which can be summed-up to determine deformations. So the relationships between actions, deformations, strains, and stresses can be used in many ways to solve the particular problems at hand. Figure 1-5 illustrates this whole process starting from loads and ending on stress resultants.

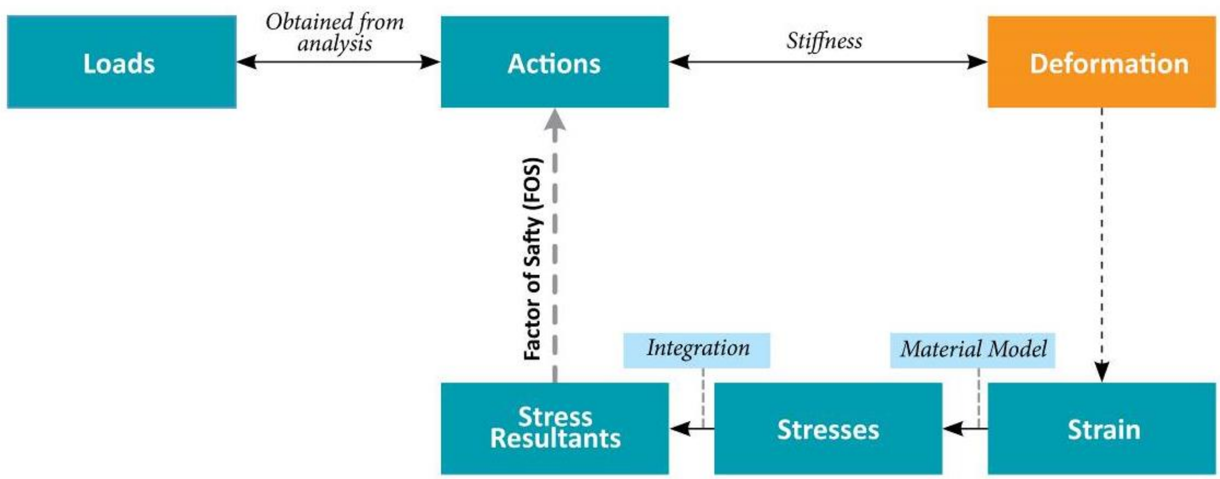


Figure 1-5: The basic relationships in structural analysis and design

A brief description of the relationship between these quantities is given here, without the explicit mathematical formulations that are adequately covered in many texts on structural theory and analysis.

a) **Action-Deformation Relationship:** Defining an action-deformation relationship means linking the deformations produced in a member due to applied actions or linking the restraining actions with applied deformations. These relationships involve the entire stiffness of the member and may be either linear or nonlinear. One action can produce more than one deformation and one deformation may be caused by more than one action.

b) **Deformation-Strain Relationship:** Deformation-Strain relationship means linking deformations with corresponding strains. Each deformation produces a particular strain pattern or profile on the cross-section. A particular strain may be result of several deformations. For example, axial strain may be produced due to axial deformation as well as flexural curvature. This relationship is defined primarily by the cross-section's stiffness and may be linear or nonlinear.

c) **Stress-Strain Relationship:** Stress-Strain relationship means linking strain to corresponding stress. Generally, this relationship is used at material level, indicating material stiffness and its behavior. For example, Hooke's Law describes the stress-strain relationship for a linear elastic material but in general, this relationship is nonlinear for most materials.

d) **Stress Resultant-Action Relationship:** This last relationship is the expression of equilibrium and completes the cycle of all relationships. In fact, this relationship is the basis for strength design of structural members, which states that "the internal stress resultants should be in equilibrium with external actions with adequate margin for safety".

1.2.1. The Concept of Stiffness

Let us consider a structure subjected to an arbitrary force (F). This force will produce an arbitrary deformation u. If we compare the structure to a simple elastic spring (Figure 1-6), a simple linear relationship between the force and deformation exists. This linear dependence is the "stiffness" which is the resistance to its deformation.

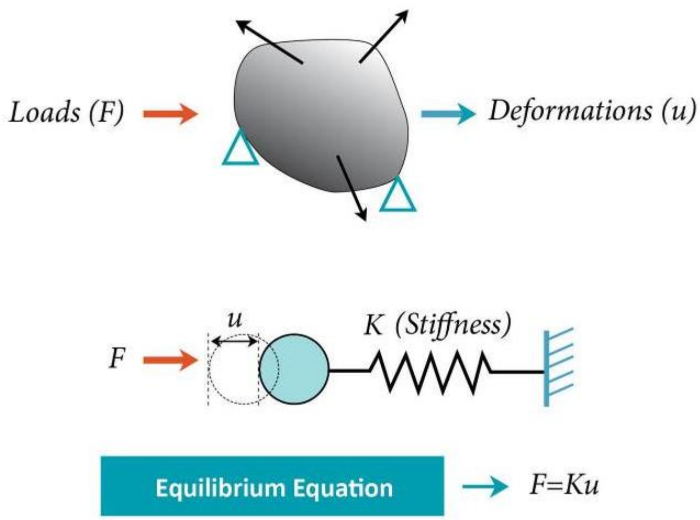


Figure 1-6: The conceptual state of equilibrium and role of stiffness

In a real structure, this resistance or stiffness comes from four sources, as shown in Figure 1-7:

a) **Global Structure Stiffness:** It is the overall resistance of the structures to overall loads and is derived from the sum of stiffness of its members, their connectivity and the boundary or the restraining conditions.
b) **Member Stiffness:** It is the resistance of each member to local actions and is derived from the cross-section stiffness and the geometry of the member.
c) **Cross-section Stiffness:** It is the resistance of the cross-section to overall strains and is derived from the cross-section geometry and the stiffness of the materials from which it is made.
d) **Material Stiffness:** It is the resistance of the material to strain and is derived from the stiffness of the material particles.

For linear elastic discrete models, the stiffness is a constant, represented by the slope of linear relationship between an "action" and the corresponding "deformation". Table 1 shows the examples of action-deformation relationships at all four levels.

Table 1: Action-deformation relationships

Sr. No.	Level	Example of an action-deformation relationship	Stiffness relating the action with corresponding deformation
1	Material Level	Stress-strain relationship	Elastic Modulus (E)
2	Cross-section Level	Moment-curvature relationship	Flexural Stiffness (EI)
3	Member Level	Moment-rotation relationship	Member Flexural Stiffness (EI/L)
4	Structural Level	Total base reaction force-roof displacement	Structural Global Stiffness Matrix ($\boldsymbol{K}$)

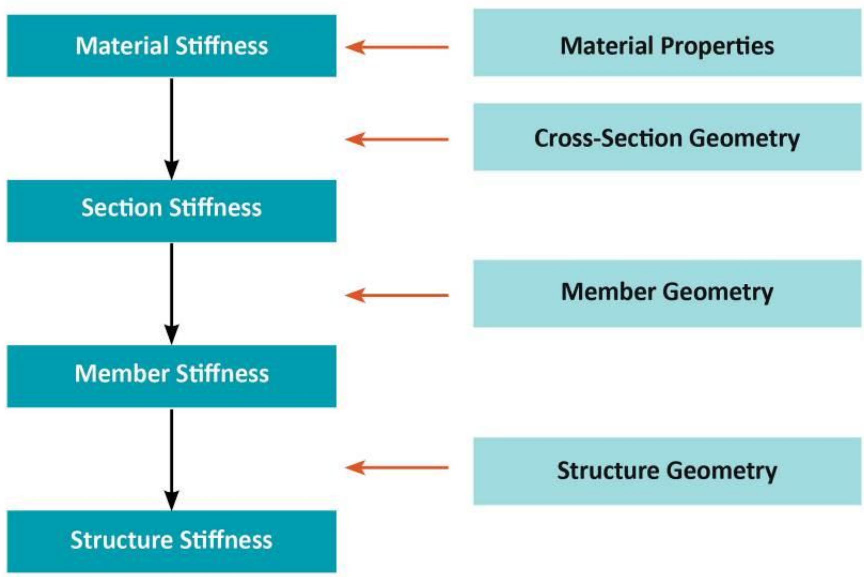


Figure 1-7: The overall stiffness of the structure is derived from the geometry and connectivity of the members and their stiffness. The member stiffness is derived from the cross-section stiffness, and member geometry. The cross-section stiffness is derived from the material stiffness and the cross-section geometry.

The simplest form of Hook's law ($F = Ku$) can be generalized to include several deformations, several actions and several stiffness relationships and represented in a matrix form. It then becomes the basis of the *"**Stiffness Matrix Method**"* of structural analysis and more generally, the *"**Finite Element Method**"*.

1.2.2. The Nonlinearity of Response and Stiffness

The equilibrium equation, $F = Ku$ is based on the assumption that the relationship between the force and deformation is linear and infinite. That means, a very large force can produce a corresponding very large displacement and an infinite force can produce an infinite displacement. The equation also suggests that if the force is decreased, the deformation will reduce and zero force will return the structure to the original un-deformed state, and that a negative force will produce exactly same negative displacement as in positive direction. However, in reality, almost none of these assumptions or behaviors are true. The relationship between force and displacement for a real structure can be highly nonlinear and inelastic with no single value of stiffness describing its behavior (Figure 1-8). In such cases, the stiffness varies at different states of deformation throughout the loading history. The complete equilibrium condition and corresponding equation should reflect not only the nonlinear and inelastic behavior, but also the effect of force being applied fast enough to produce deformation with velocity and acceleration so that the total equilibrium should include effect of inertia and damping.

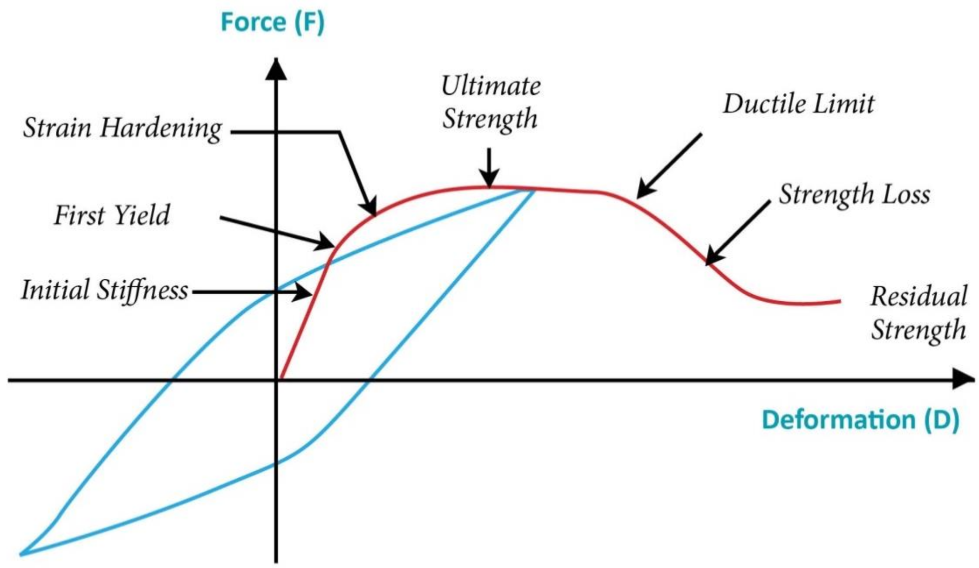


Figure 1-8: A typical relationship between force and deformation. This may also hold for relationship between stress and strain

Figure 1-9 shows all possible states of equilibrium for a structure whereas Figure 1-10 shows the nonlinearity inherent in various stages of stiffness contribution.

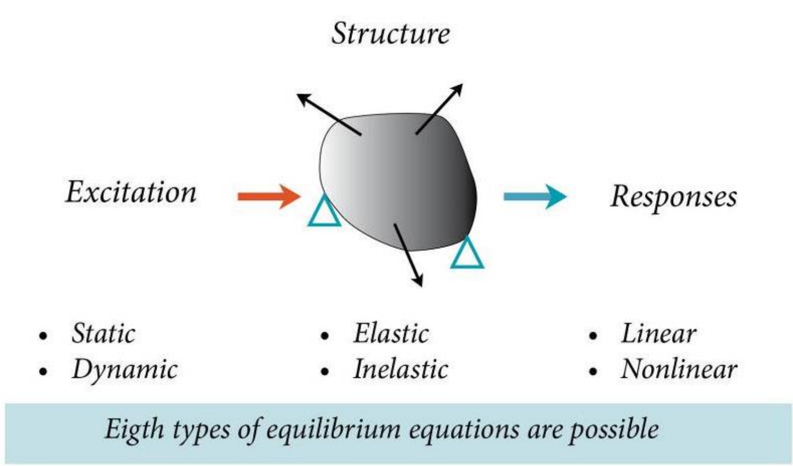


Figure 1-9: The equilibrium conditions for a typical structural system

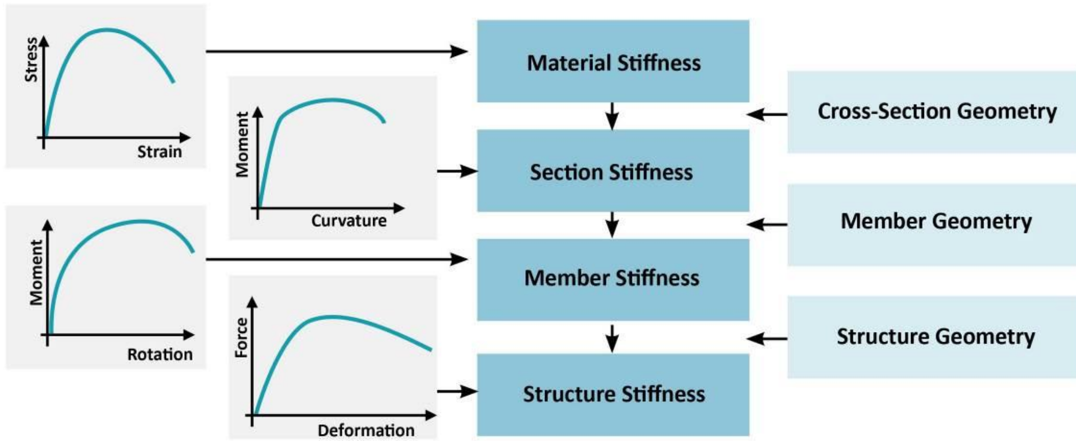


Figure 1-10: The nonlinear nature of stiffness and the role of nonlinear cross-sectional response

1.3. The Concept of Degree of Freedom (DOF)

Although it is possible to conceive situations where deformation can occur without external actions (such as thermal variation) generally, an external action (a generalized force, moment, or torque) is needed to produce deformation in a structural member. If the actions can be generalized in terms of their components, we can say that in general, those actions components produce corresponding deformation components. If we assume that the materials are behaving linearly and elastically, we can end up with a simple spring representation for each deformation component. That is, the action (F) and deformation (u) in a particular sense are proportional and related to each other by the corresponding stiffness.

Consider, for example, the cross-section of a beam member (broadly defined as a structural component having one dimension significantly larger than the other two) shown in Figure 1-11. The longer dimension becomes the member axis, and the dimensions in the plane perpendicular to the longitudinal axis define the cross-section. The cross-section exposes the materials used in the member.

For a member placed in a general three-dimensional space, each point in a member can move in an infinite number of ways. However, if the cross-section is assumed to be rigid in its own plane, all these movements can be completely defined in terms of seven idealized directions, referred to as *the degrees of freedom (DOF)* at each section on the centerline of the member, with respect to three orthogonal axes.

Using the right hand rule, if we orient the axis system as shown in the Figure 1-11, these degrees of freedom become:

- Movement along the member axis, u_z
- Movement along the x- axis, u_x
- Movement along the y-axis, u_y
- Rotation about the longitudinal axis, r_z
- Rotation about the x-axis, r_x
- Rotation about the y-axis, r_y
- Out-of-plane movement (distortion) of the cross-section's points along the longitudinal axis, w_z

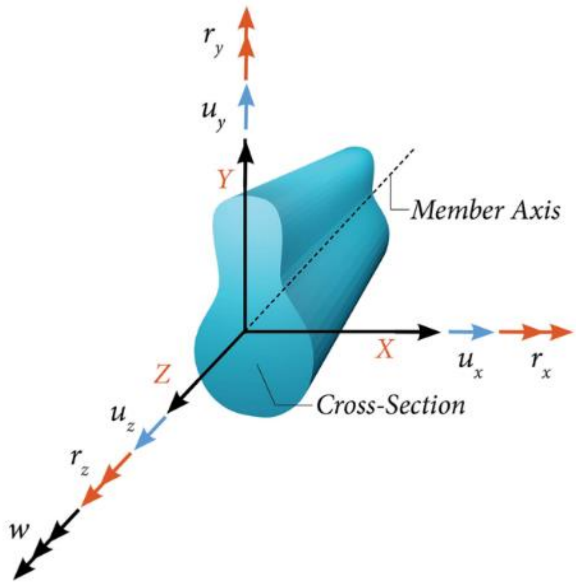


Figure 1-11: Degrees of freedom. Each Section on a beam member can have seven Degrees of Freedom (DOF) with respect to its local axis

In short, the number of independent coordinates necessary to specify the configuration or position of a system at any time is referred to as the *number of degrees of freedom*. In general, a continuous structure has an infinite number of degrees of freedom. Nevertheless, the process of idealization or selection of an appropriate mathematical model permits the reduction in the number of degrees of freedom to a discrete number and in some cases to just a single degree of freedom.

It is important to note that in the above definitions, the cross-section is assumed to be rigid in its own plane. That means the dimensions and shape of the cross-section before and after deformation remain the same. This assumption is mostly true for solid sections. For thin walled sections and for large box girders, the section may distort in its own plane and some additional considerations may be needed to evaluate its behavior. It is also generally assumed that the member centerline passes through the geometric (or the plastic) centroid of the cross-section. This assumption is generally true for members having length more than at least five times the average size of the cross-section.

1.3.1. Degrees of Freedom, Deformations, Strains and Stresses

Each degree of freedom at the cross-section centroid is associated with a corresponding deformation in the member. Each deformation in the member produces a corresponding strain profile in the cross-section. Each strain profile generally produces a corresponding stress profile in the cross-section material(s). These relationships are shown below.

u_z → Axial deformation → Axial strain → Axial stress

u_x → Shear deformation → Shear strain → Shear stress

u_y → Shear deformation → Shear strain → Shear stress

r_z → Torsion → Shear strain → Shear stress (may also produce axial stresses and strains)

r_y → Curvature → Axial strain → Axial stress (may also produce shear strains)

r_x → Curvature → Axial strain → Axial stress (may also produce shear stresses and strains)

w_z → Warping → Axial strain → Axial stress (may also produce shear strains)

Sometimes one DOF may be related to more than one stress and strain component. In some cases, strain may not produce any stress, such as the unrestrained thermal expansion or free shrinkage produces elongation and corresponding strains, but does not result in any stresses in the cross-section's material.

"Each deformation may produce more than one strain and stress component and each stress component may be produced by more than one deformation."

1.3.2. Internal Stress Resultants and Degrees of Freedom

Material stresses in the cross-section (or stress components along the reference axes) can be summed up to obtain the total resultants. These stress resultants (as shown in Figure 1-12), when determined with respect to the member axes and the corresponding degrees of freedom at the cross-section centroid can provide useful information related to the "capacity" of the cross-section.

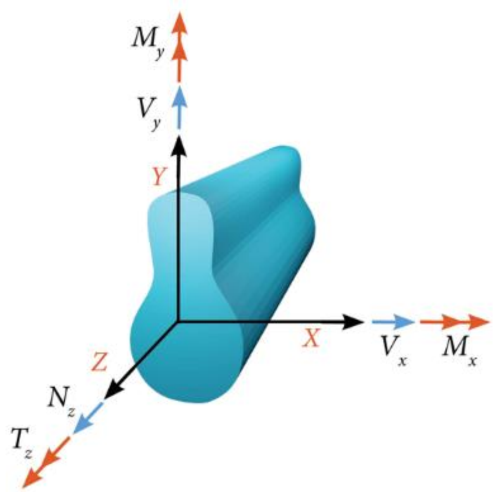


Figure 1-12: Stress resultants and degrees of freedom

1.4. Different Forms of Linear Stiffness Relationships

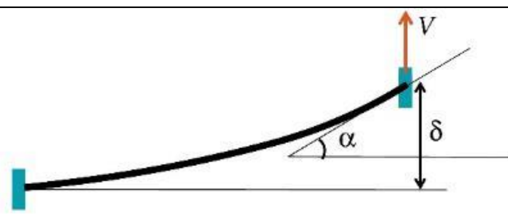
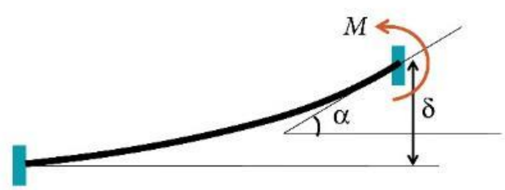
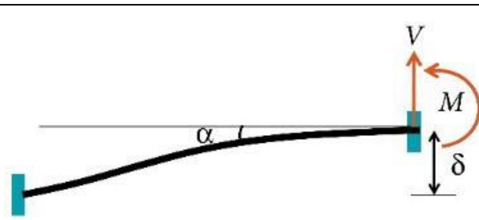
Consider a single beam-type member with six degrees of freedom at each end (ignoring warping). Two different types of action-deformation relationships can be derived for this example. In the first case, the total deformations in a particular degree of freedom can be computed from all actions that contribute towards this deformation, assuming that all non-participating DOFs are locked. The second type of relationships can be used to compute the restraining actions needed to prevent deformations in all contributing DOFs, while all other DOFs are assumed to be locked. These two approaches are the basis of the *Flexibility* and *Stiffness Matrix*, (or *Force and Displacement Methods*) for structural analysis, respectively.

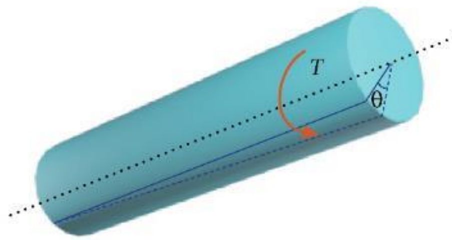
1.4.1. Deformations for Applied Actions: Flexibility Relationships

For linear elastic elements, it is possible to develop first type of relationships (relating the applied actions with corresponding degree-of-freedom deformations) using principles of mechanics of materials. All we need to know is the stiffness quantity relating each action-deformation pair. For a simple beam element with applied actions on its right end (and assuming the left end fully restrained), the three possible actions are shown in Figure 1-13. These are the axial load (P), shear force (V) and bending moment (M), corresponding to three assumed degrees of freedom on its right end. Table 1-1 shows the deformations in this beam member for few cases of applied actions.

Figure 1-13: Three possible “actions” corresponding to deformations on three DOFs on right end

Table 1-2: Some linear elastic flexibility relationships for a simple beam element

Case	Illustration	Flexibility Relationships
Axial deformation Δ under lateral force P		$\Delta = \dfrac{PL}{AE}$
Vertical deformation “δ” and rotation “α” under vertical force “V” only		$\delta = \dfrac{VL^3}{3EI}$ $\alpha = \dfrac{VL^2}{2EI}$
Vertical displacement “δ” and rotation “α” under Moment (M) only		$\delta = \dfrac{ML^2}{2EI}$ $\alpha = \dfrac{ML}{EI}$
Vertical displacement “δ” and rotation “α” under combined shear force (V) and Moment (M)		$\delta = \dfrac{L^3}{6EI}\left(2V - \dfrac{3M}{L}\right)$ $\alpha = \dfrac{L^2}{2EI}\left(\dfrac{2M}{L} - V\right)$

The rotation along the member (θ) due to torque (T) applied at the ends, excluding the effect of warping		$\theta = \dfrac{T\,L}{GJ}$

It is important to note that in all flexibility relationships, actions are related to deformations by various cross-sectional properties (e.g. A, I and J), material properties (e.g. G and E) and length of member (L).

1.4.2. Restraining Actions for Assumed Deformations: Stiffness Relationships

For the derivation and development of the stiffness matrix and finite element methods for structural analysis, it is often convenient to develop second type of relationships (involving assumed deformations and the restraining actions needed to “prevent” that deformation). These are actually inverse of the relationships that are used to compute deformations for applied actions. For the same example beam element, the restraining actions against assumed deformations for few common cases are shown in table 1-2.

Table 1-3: Some restraining actions related by corresponding deformations through linear elastic stiffness

Case	Actions for assumed deformations: Stiffness relationships
Axial force P due to axial deformation Δ	$P = \dfrac{E\,A}{L}\Delta$
Shear force V at the restraining end for deflection “v” and rotation “α” at the other end	$V = \dfrac{12E\,I}{L^3}v + \dfrac{6E\,I}{L^2}\alpha$
Moment M at the restraining end for deflection “v” and rotation “α” at the other end	$M = \dfrac{6E\,I}{L^2}v + \dfrac{4E\,I}{L}\alpha$
Restraining torque T due to axial rotation θ	$T = \dfrac{G\,J}{L}\theta$

1.5. Static vs. Dynamic Problems

A structural dynamic problem differs from its static loading counterpart in two important respects.

The first difference to be noted, by definition, is the time-varying nature of the dynamic problem. Because both loading and response vary with time, it is evident that a dynamic problem does not have a single solution, as a static problem does; instead the analyst must establish a succession of solutions corresponding to all times of interest in the response history. Thus a dynamic analysis is clearly more complex and time consuming than a static analysis.

The second and more fundamental distinction between static and dynamic problems is the occurrence of inertial forces due to non-zero accelerations in dynamic problems. For example, if a simple beam is subjected to a static load p, its internal moments and shears and deflected shape depend only upon this load and they can be computed by established principles of force equilibrium (Figure 1-14(a)). On the other hand, if the load p(t) is applied dynamically, the resulting displacements of the beam depend not only upon this load but also upon inertial forces which oppose the accelerations producing them (Figure 1-14(b)). Thus the corresponding internal moments and shears in the beam must equilibrate not only the externally applied force p(t) but also the inertial forces resulting from the accelerations of the beam.

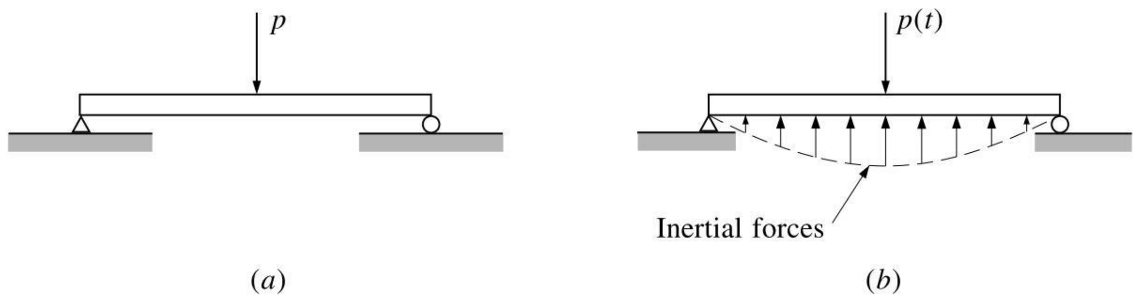


Figure 1-14: Basic difference between static and dynamic loads: (a) static loading; (b) dynamic loading. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

Inertial forces which resist accelerations of the structure in this way are the most important distinguishing characteristic of a structural dynamics problem. In general, if the inertial forces represent a significant portion of the total load equilibrated by the internal elastic forces of the structure, then the dynamic character of the problem must be accounted for in its solution. On the other hand, if the motions are so slow that the inertial forces are negligibly small, the analysis of response for any desired instant of time may be made by static structural analysis procedures even though the load and response may be time-varying.

1.6. Classification of Structural Models

A structural model is a mathematical description of physical structure. It provides the link between the real physical system and the mathematically feasible solution. It is the symbolic designation for the substitute idealized system including all the assumptions imposed on the physical problem.

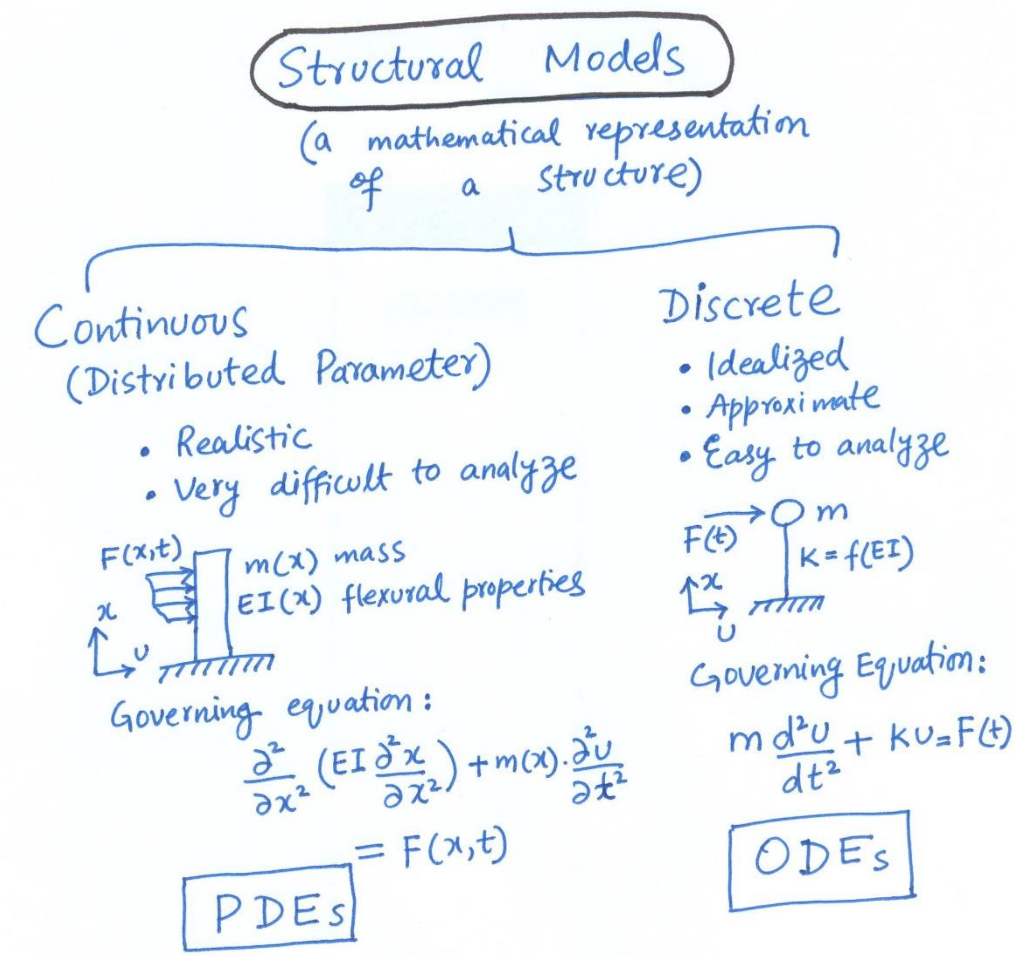


Figure 1-15: Continuous vs. discrete structural models

1.7. Discretization of Structures

The mass of a real structural component (say a beam) is distributed continuously along its length, the displacements and accelerations must be defined for each point along the axis if the inertial forces are to be completely defined. In this case, the analysis must be formulated in terms of partial differential equations because position along the span as well as time must be taken as independent variables. However, the analytical problem is greatly simplified by discretizing the structure by any of the ways explained below.

1.7.1. Lumped Mass Procedure

In lumped mass procedure, the mass of the structure is assumed to be concentrated at discrete points. The analytical solution becomes greatly simplified because inertial forces develop only at these mass points. In this case, it is necessary to define the displacements and accelerations only at these discrete locations.

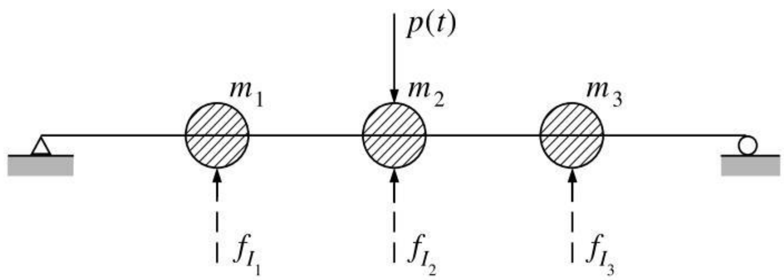


Figure 1-16: Lumped-mass idealization of a simple beam. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

If the three masses in the system of Figure 1-16 are fully concentrated and are constrained so that the corresponding mass points translate only in a vertical direction, this would be called a three-degree-of-freedom (3 DOF) system. On the other hand, if these masses are not fully concentrated so that they possess finite rotational inertia, the rotational displacements of the three points will also have to be considered, in which case the system has 6 DOF. If axial distortions of the beam are significant, translation displacements parallel with the beam axis will also result giving the system 9 DOF.

More generally, if the structure can deform in three-dimensional space, each mass will have 6 DOF; then the system will have 18 DOF. However, if the masses are fully concentrated so that no rotational inertia is present, the three-dimensional system will then have 9 DOF. On the basis of these considerations, it is clear that a system with continuously distributed mass has an infinite number of degrees of freedom.

1.7.2. Generalized Displacement Procedure

The lumped-mass idealization described above provides a simple means of limiting the number of degrees of freedom that must be considered in conducting a dynamic analysis of an arbitrary structural system. The lumping procedure is most effective in treating systems in which a large proportion of the total mass actually is concentrated at a few discrete points. Then the mass of the structure which supports these concentrations can be included in the lumps, allowing the structure itself to be considered weightless.

However, in cases where the mass of the system is quite uniformly distributed throughout, an alternative approach to limiting the number of degrees of freedom may be preferable. This procedure is based on the assumption that the deflected shape of the structure can be expressed as the sum of a series of specified displacement patterns; these patterns then become the displacement coordinates of the structure. A simple example of this approach is the trigonometric-series representation of the deflection of a simple beam. In this case, the deflection shape may be expressed as the sum of independent sine-wave contributions, as shown in Figure 1-17, or in mathematical form,

$$v(x) = \sum_{n=1}^{\infty} \frac{b_n n\pi x}{L}$$

In general, any arbitrary shape compatible with the prescribed support conditions of the simple beam can be represented by this infinite series of sine-wave components. The amplitudes of the sine-wave shapes may be considered to be the displacement coordinates of the system, and the infinite number of degrees of freedom of the actual beam are represented by the infinite number of terms included in the series. The advantage of this approach is that a good approximation to the actual beam shape can be achieved by a truncated series of sine-wave components; thus a 3 DOF approximation would contain only three terms in the series, etc.

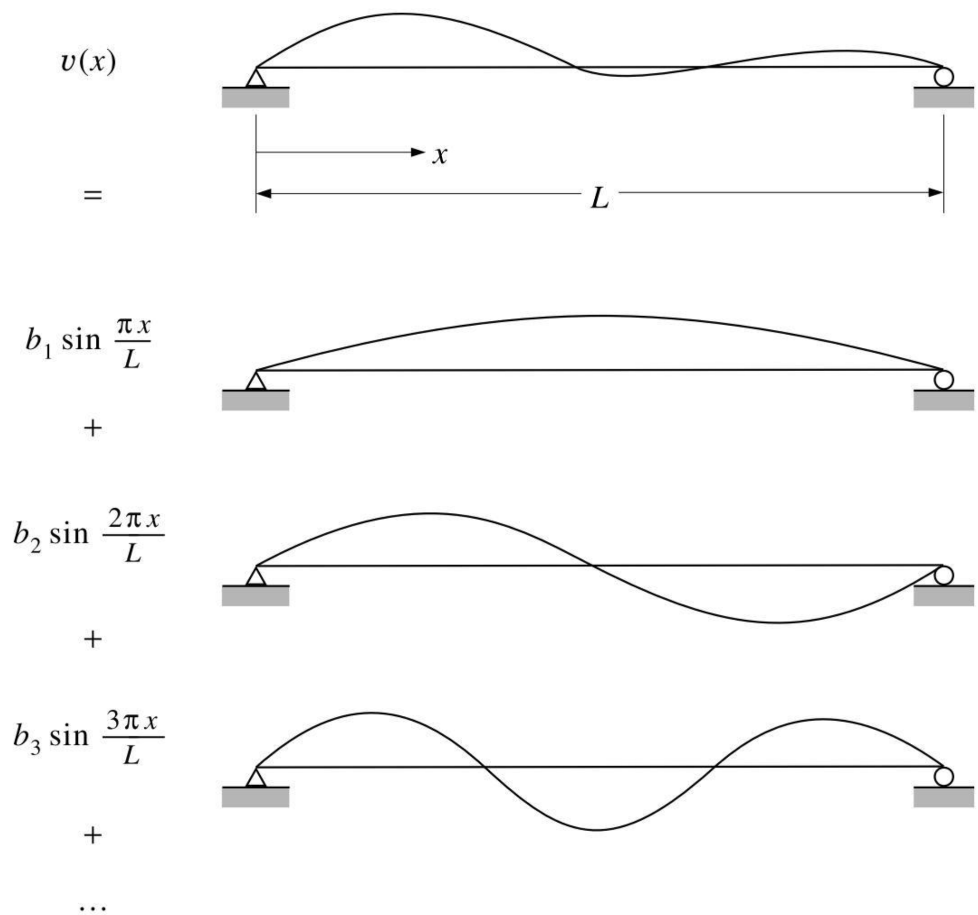


Figure 1-17: Sine-series representation of simple beam deflection. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

This concept can be further generalized by recognizing that the sine-wave shapes used as the assumed displacement patterns were an arbitrary choice in this example. In general, any shapes ψ n (x) which are compatible with the prescribed geometric-support conditions and which maintain the necessary continuity of internal displacements may be assumed. Thus a generalized expression for the displacements of any one-dimensional structure might be written

$$v(x) = \sum_n Z_n \psi_n(x)$$

For any assumed set of displacement functions ψ(x), the resulting shape of the structure depends upon the amplitude terms Z n , which will be referred to as generalized coordinates. The number of assumed shape patterns represents the number of degrees of freedom considered in this form of idealization. In general, better accuracy can be achieved in a dynamic analysis for a given number of degrees of freedom by using the shape-function method of idealization rather than the lumped-mass approach. However, it also should be recognized that greater computational effort is required for each degree of freedom when such generalized coordinates are employed. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

1.7.3. Finite Element Concept

A third method of expressing the displacements of any given structure in terms of a finite number of discrete displacement coordinates, which combines certain features of both the lumped-mass and the generalized-coordinate procedures, has now become popular. This approach, which is the basis of the finite-element method of analysis of structural continua, provides a convenient and reliable idealization of the system and is particularly effective in digital-computer analyses.

The finite-element type of idealization is applicable to structures of all types: framed structures, which comprise assemblages of one-dimensional members (beams, columns, etc.); plane-stress, plate- and shell-type structures, which are made up of two-dimensional components; and general three-dimensional solids. For simplicity, only the one-dimensional type of structural components will be considered in the present discussion, but the extension of the concept to two- and three-dimensional structural elements is straightforward.

The first step in the finite-element idealization of any structure, e.g., the beam shown in Figure 1-18, involves dividing it into an appropriate number of segments, or elements, as shown. Their sizes are arbitrary; i.e., they may be all of the same size or all different. The ends of the segments, at which they are interconnected, are called nodal points. The displacements of these nodal points then become the generalized coordinates of the structure.

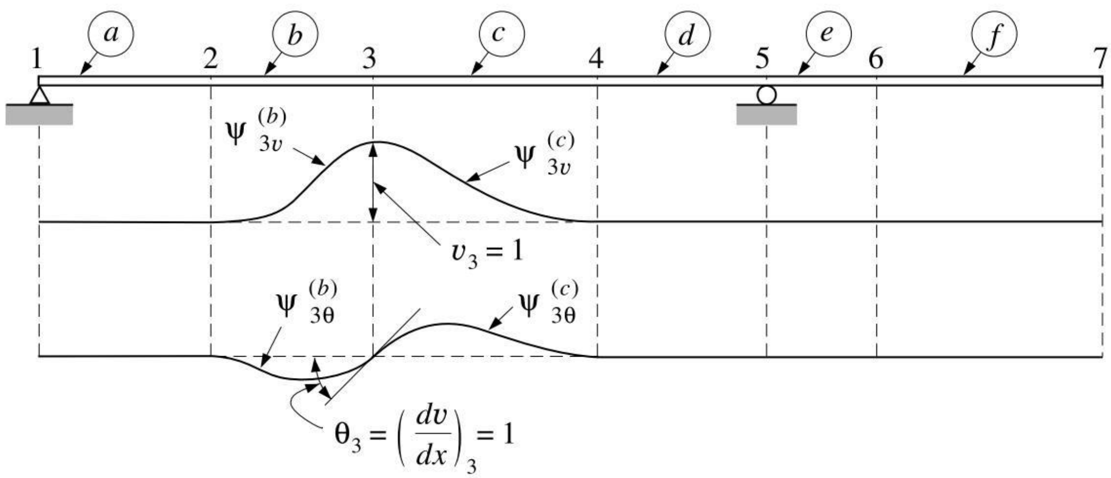


Figure 1-18: Typical finite-element beam coordinates. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

The deflection shape of the complete structure can now be expressed in terms of these generalized coordinates by means of an appropriate set of assumed displacement functions using an expression similar to Eq. (1-2). In this case, however, the displacement functions are called interpolation functions because they define the shapes produced by specified nodal displacements. For example, Figure 1-18 shows the interpolation functions associated with two degrees of freedom of nodal point 3, which produce transverse displacements in the plane of the figure. In principle, each interpolation function could be any curve which is internally continuous and which satisfies the geometric displacement condition imposed by the nodal displacement. For one-dimensional elements it is convenient to use the shapes which would be produced by these same nodal displacements in a uniform beam. It will be shown later in Chapter 10 that these interpolation functions are cubic hermitian polynomials.

Because the interpolation functions used in this procedure satisfy the requirements stated in the preceding section, it should be apparent that coordinates used in the finite-element method are just special forms of generalized coordinates. The advantages of this special procedure are as follows:

a) The desired number of generalized coordinates can be introduced merely by dividing the structure into an appropriate number of segments.
b) Since the interpolation functions chosen for each segment may be identical, computations are simplified.
c) The equations which are developed by this approach are largely uncoupled because each nodal displacement affects only the neighboring elements; thus the solution process is greatly simplified.

In general, the finite-element approach provides the most efficient procedure for expressing the displacements of arbitrary structural configurations by means of a discrete set of coordinates. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

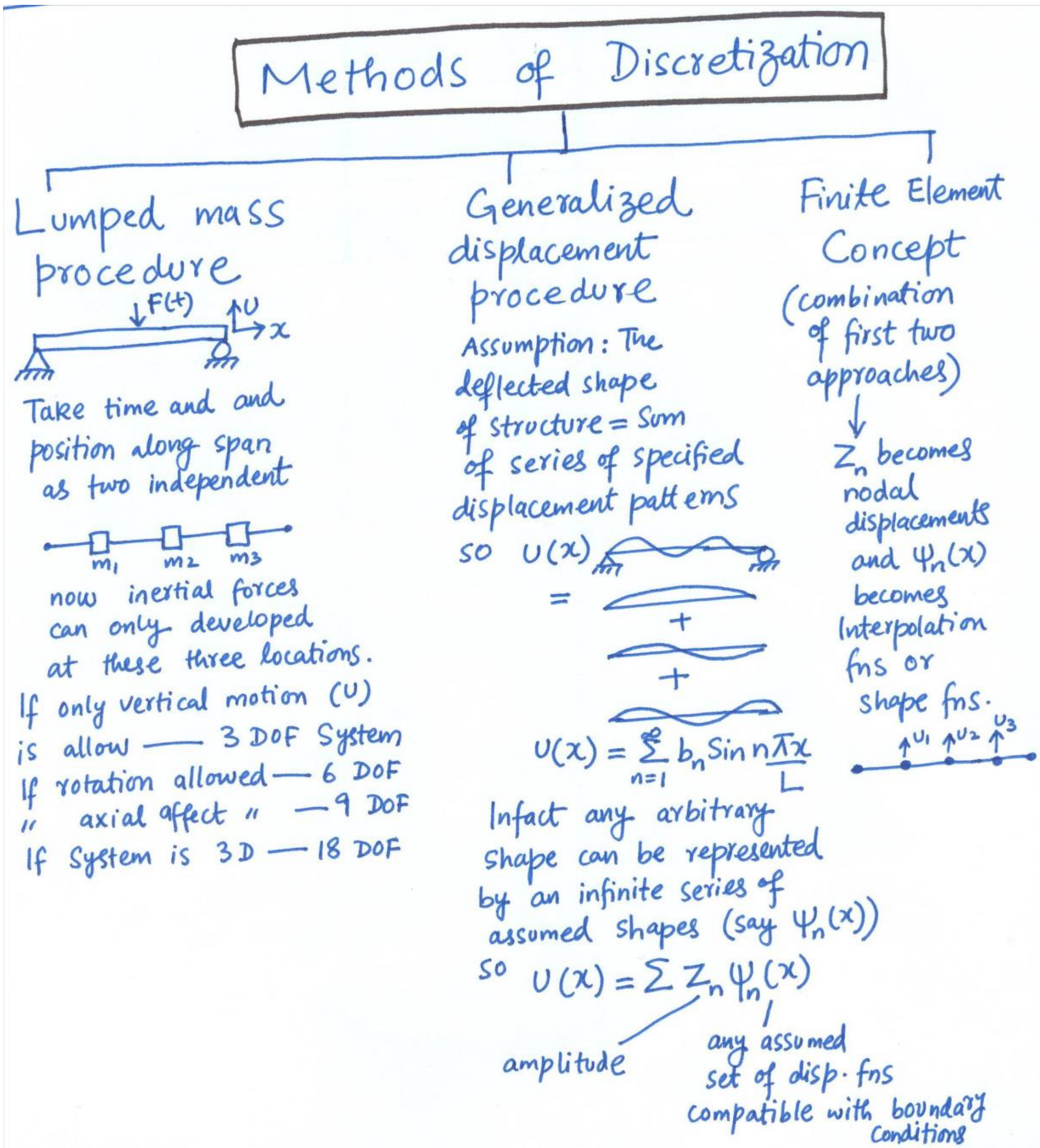
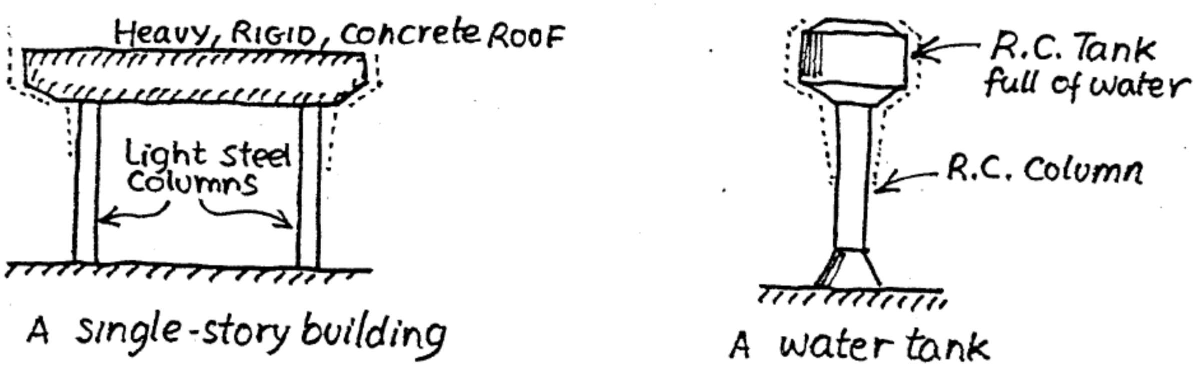
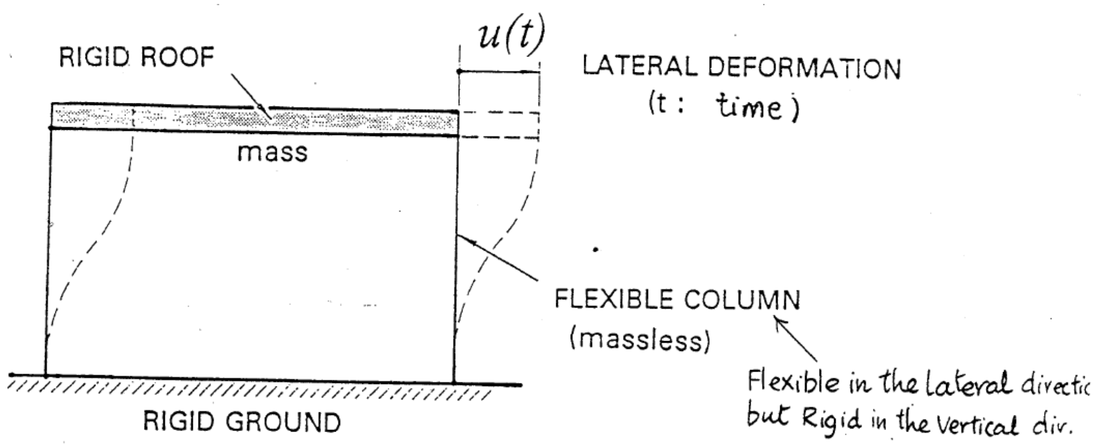


Figure 1-19: Three methods of discretization



(a) Examples of simple structures



(b) An idealized model of simple structures subjected to a dynamic excitation

Figure 1-20: Lumped mass and lumped stiffness idealization of a simple structure

1.8. Equations of Motion

The mathematical expressions defining the dynamic displacements are called the equations of motion of the structure, and the solution of these equations of motion provides the required displacement time histories. The formulation of the governing equations of motion is possibly the most important phase of the entire analysis procedure (and sometimes also the most difficult phase).

There are three ways to formulate the equations of motion.

a) Direct dynamic equilibration
b) Principle of virtual work
c) Variational Approach, Lagrange's equations (Hamilton's principle)

1.8.1. Direct Equilibrium using D'Alembert's Principle

Consider a system of dynamic forced applied to a mass m as shown in Figure 1-21.

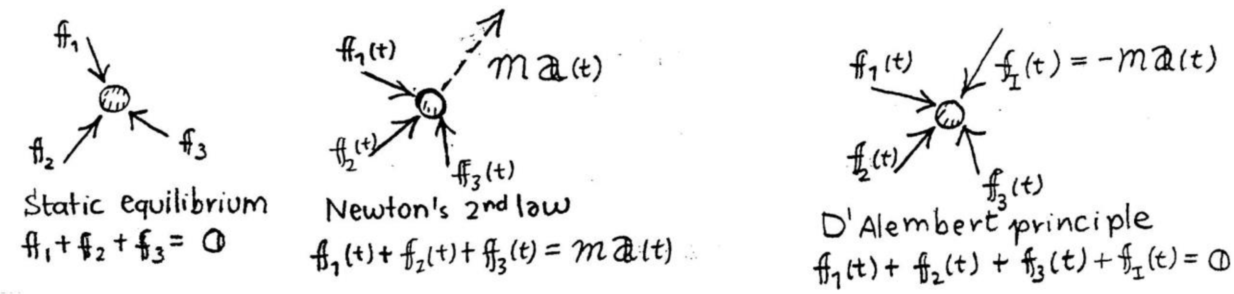


Figure 1-21: A mass m subjected to a system of dynamic forces

$\boldsymbol{f_1}, \boldsymbol{f_2}, \boldsymbol{f_3}$ are applied forces vectors. $\boldsymbol{a}$ is the acceleration of particle mass m.

Newton's 2nd law states that, "The rate of change of momentum of any mass m is equal to the force acting on it". Therfore,

$$\boldsymbol{f_1}(t) + \boldsymbol{f_2}(t) + \boldsymbol{f_3}(t) = \frac{d}{dt}\left(m.\frac{d\boldsymbol{r}(t)}{dt}\right) = m\frac{d^2\boldsymbol{r}}{dt^2} = m\boldsymbol{a}$$

D'Alembert's concept states that "A mass develops an inertia force in proportion to its acceleration and opposing it". Therfore,

$\boldsymbol{f_1}(t) + \boldsymbol{f_2}(t) + \boldsymbol{f_3}(t) + \boldsymbol{f_I}(t) = \boldsymbol{0}$ where $\boldsymbol{f_I}(t) = -m\boldsymbol{a}(t)$

$$\sum \boldsymbol{F}(t) = \boldsymbol{0}$$

All dynamic forces are in equilibrium— Dynamic Equilibrium (including inertia force)

This is a very convenient concept structure dynamics because its permits equations of motion to be expressed as of as "equations of dynamic equilibrium".

1.8.2. Principle of Virtual Work

If the structural system is reasonably complex involving a number of interconnected mass points or bodies of finite size, the direct equilibration of all the forces acting in the system may be difficult. Frequently, the various forces involved may readily be expressed in terms of the displacement degrees of freedom, but their equilibrium relationships may be obscure. In this case, the principle of virtual displacements can be used to formulate the equations of motion as a substitute for the direct equilibrium relationships.

The principle of virtual displacements may be expressed as follows. If a system which is in equilibrium under the action of a set of externally applied forces is subjected to a virtual displacement, i.e., a displacement pattern compatible with the system's constraints, the total work done by the set of forces will be zero. The statement can be written as,

"For a deformable body in equilibrium under a set of forces and moments, the sum of virtual work (internal and external) is zero. Virtual work is the work done by forces and moments under virtual displacements."

$$\sum \delta W = 0$$

With this principle, it is clear that the vanishing of the work done during a virtual displacement is equivalent to a statement of equilibrium. Thus, the response equations of a dynamic system can be established by first identifying all the forces acting on the masses of the system, including inertial forces defined in accordance with D'Alembert's principle. Then, the equations of motion are obtained by separately introducing a virtual displacement pattern corresponding to each degree of freedom and equating the work done to zero. A major advantage of this approach is that the virtual work contributions are scalar quantities and can be added algebraically, whereas the forces acting on the structure are vectorial and can only be superposed vectorially. (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

1.8.3. Vriational Approach

Another means of avoiding the problems of establishing the vectorial equations of equilibrium is to make use of scalar quantities in a variational form known as Hamilton's principle. Inertial and elastic forces are not explicitly involved in this principle; instead, variations of kinetic and potential energy terms are utilized. This formulation has the advantage of dealing only with purely scalar energy quantities, whereas the forces and displacements used to represent corresponding effects in the virtual›work procedure are all vectorial in character, even though the work terms themselves are scalars.

It is of interest to note that Hamilton's principle can also be applied to statics problems. In this case, it reduces to the well-known principle of minimum potential energy so widely used in static analyses.

Note:

The equation of motion of a dynamic system can be formulated by any one of three distinct procedures. The most straightforward approach is to establish directly the dynamic equilibrium of all forces acting in the system, taking account of inertial effects by means of d'Alembert's principle. In more complex systems, however, especially those involving mass and elasticity distributed over finite regions, a direct vectorial equilibration may be difficult, and work or energy formulations which involve only scalar quantities may be more convenient. The most direct of these procedures is based on the principle of virtual displacements, in which the forces acting on the system are evaluated explicitly but the equations of motion are derived by consideration of the work done during appropriate virtual displacements.

On the other hand, the alternative energy formulation, which is based on Hamilton's principle, makes no direct use of the inertial or conservative forces acting in the system; the effects of these forces are represented instead by variations of the kinetic and potential energies of the system. It must be recognized that all three procedures are completely equivalent and lead to identical equations of motion. The method to be used in any given case is largely a matter of convenience and personal preference; the choice generally will depend on the nature of the dynamic system under consideration. (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

1.9. Components of a Discrete Dynamic System

If the roof of a simple structure is displaced laterally by a distance u_o and then released, the idealized structure will oscillate around its initial equilibrium configuration as shown in the Figure 1-21. The displacement history shows the amplitude of roof displacement as a function of time.

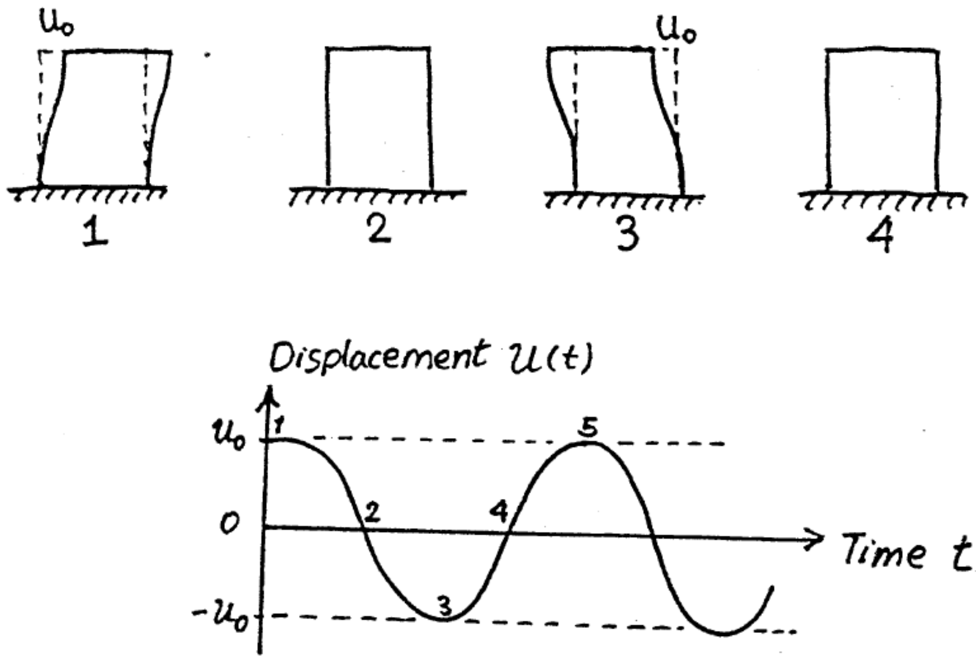


Figure 1-21: The oscillation of a structure with amplitude u_o

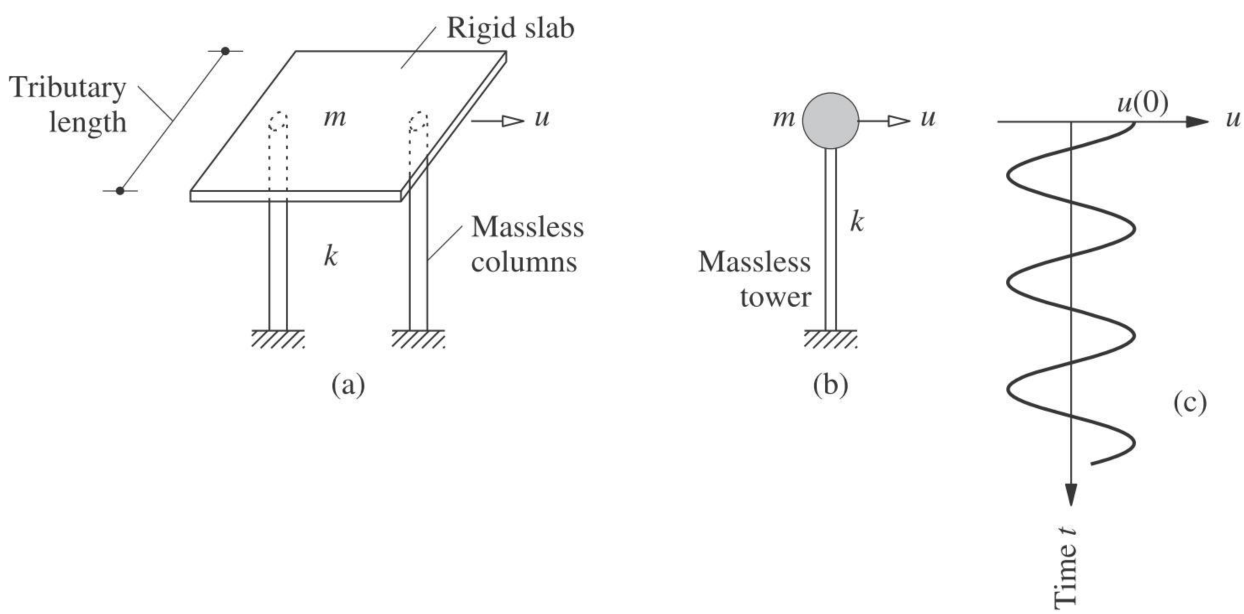


Figure 1-22: (a) Idealized pergola; (b) idealized water tank; (c) free vibration due to initial displacement. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

Figure 1-22 also shows an example of an idealized pergola and an idealized water tank. The displacement history (Figure 1-22 (c)) shows the amplitude of free vibration due to initial displacement as a function of time (Taken from Chopra (2012) Dynamics of Structures, 4th Edition).

The oscillation in these examples will continue with the same amplitude u_o and the idealized structure will never come to rest. This is an unrealistic response because the actual structure will oscillate with decreasing amplitude and will eventually come to rest.

To incorporate this feature into the idealized structure, an energy dissipating mechanism is required. Therefore, an energy absorbing element is introduced in the idealized structure which is called the viscous damping element (denoted by a dashpot). Hence, a dynamic system has three important components, as discussed below.

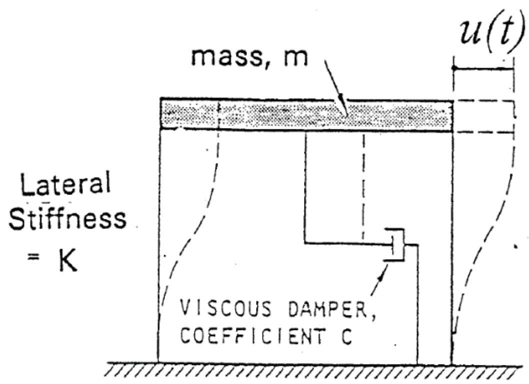


Figure 1-23: The functional elements of a simple (discrete parameter) dynamic system. Many basic concepts in structural dynamics can be understood by studying this simple structure (also sometimes called a single-degree-of-freedom system)

1.9.1. Mass and Inertial Force

D'Alembert's principle states that the mass develops an "inertial force" proportional to its acceleration in an opposing direction ($F = ma$).

Similarly, Newton;s 2nd law states that the rate of change of momentum of mass is equal to the force acting on it.

$$p(t) = \frac{\partial}{\partial t}\left(m\frac{\partial u}{\partial t}\right)$$

Where $m\frac{\partial u}{\partial t}$ is the momentum. Therefore, the mass of the structure and the resulting inertial force (f_I) is an essential component of a dynamic system.

$$f_I = m\ddot{u}$$

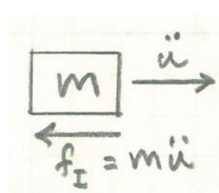


Figure 1-24: Component 1 - The mass and inertial force

1.9.2. Lateral Stiffness and Elastic Restoring Force

The elastic lateral stiffness of a structure (k) and the resulting elastic restoring force (f_s) is another important component of a dynamic system. For static analysis, only this component is considered. The lateral stiffness is generally modeled as a "spring".

$$f_s = ku$$

The area under the graphical relationship between elastic force f_s and the structure's displacement u is a measure of the elastic potential energy (E_s) stored in the structure.

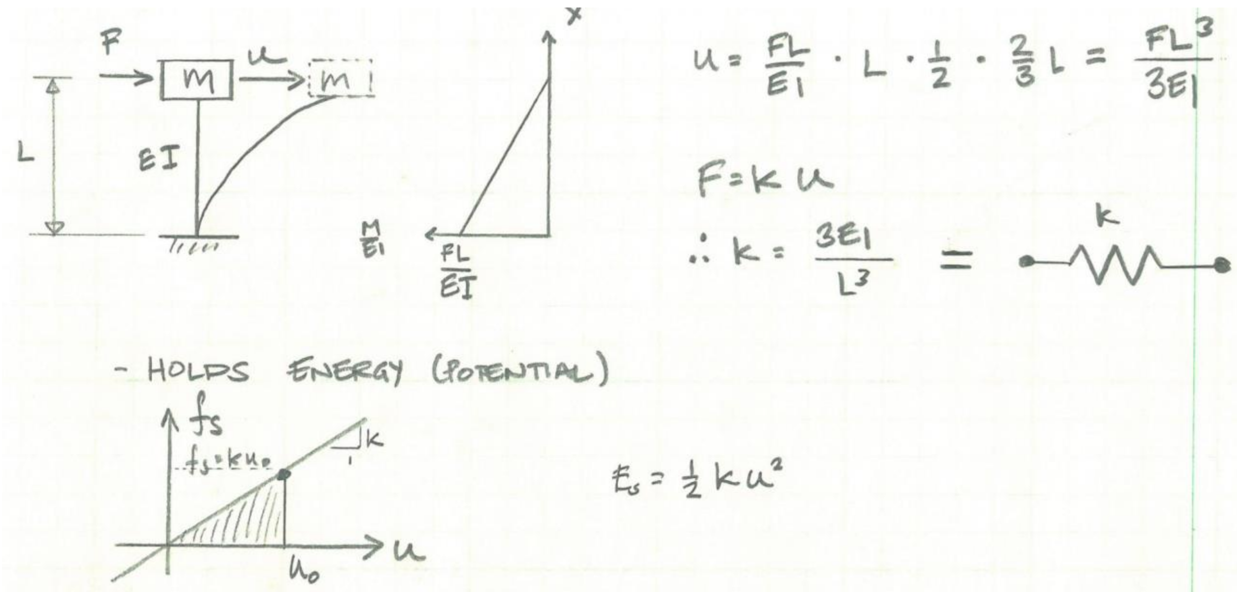


Figure 1-25: Component 2 - The elastic stiffness and elastic restoring force

Sometimes it is necessary to determine the equivalent spring constant for a system in which two or more springs are arranged in parallel as shown in Figure 1-26 (a) or in series as in Figure 1-26 (b).

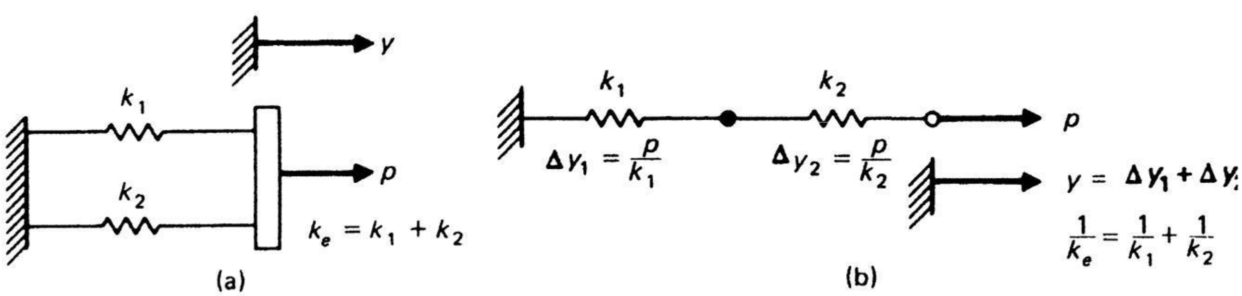


Figure 1-26: Combination of springs. (a) Springs in parallel. (b) Springs in series. (Mario Paz (2003) Structural Dynamics: Theory and Computation, 5th Edition)

For two springs in parallel the total force required to produce a relative displacement of their ends of one unit is equal to the sum of their spring constants. This total force is by definition the equivalent spring constant k_e and is given by

$$k_e = k_1 + k_2$$

In general for n springs in parallel,

$$k_e = \sum_{i=1}^{n} k_i$$

For two springs assembled in series as shown in Figure 1-26 (b), the force p produces the relative displacements in the springs,

$$\Delta y_1 = p/k_1 \text{ and } \Delta y_2 = p/k_2$$

Then, the total displacement y of the free end of the spring assembly is equal to $y = \Delta y_1 + \Delta y_2$ or substituting Δy_1 and Δy_2,

$$y = p/k_1 + p/k_2$$

Consequently, the force necessary to produce one unit displacement (equivalent spring constant) is given by

$$k_e = p/y$$

Substituting y from this last relation into above equation, we may conveniently express the reciprocal value of the equivalent spring constant as

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

In general, for n springs in series the equivalent spring constant may be obtained from,

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i}$$

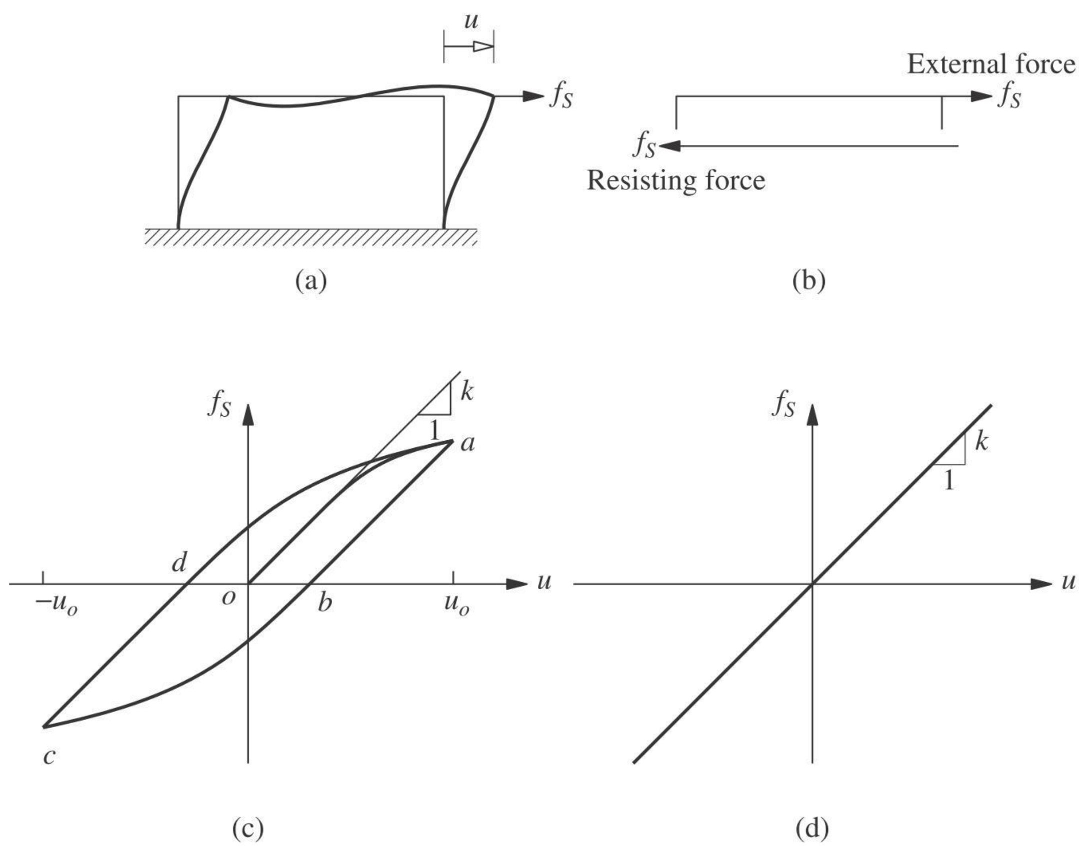


Figure 1-27: (a) A portal frame subjected to an external force f_s, (b) The resisting force would be equal and in opposite direction to f_s, (c) A nonlinear relation between restoring force and displacement, and (d) A linear relationship between lateral restoring force and displacement (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

Note: Shear Behavior vs. Flexural Behavior of Frame:

Consider the frame of with bay width L, height h, elastic modulus E, and second moment of the cross-sectional area (or moment of inertia) about the axis of bending = I_b and I_c for the beam and columns, respectively; the columns are clamped (or fixed) at the base. The lateral stiffness of the frame can readily be determined for two extreme cases: If the beam is rigid [i.e., flexural rigidity $E\ I_b\ =\ \infty$

$$k = \sum_{Columns} \frac{12EI_c}{h^3} = \frac{24EI_c}{h^3}$$

On the other hand, for a beam with no stiffness [i.e., $EI_b\ =\ 0$]

$$k = \sum_{Columns} \frac{3EI_c}{h^3} = \frac{6EI_c}{h^3}$$

Observe that for the two extreme values of beam stiffness, the lateral stiffness of the frame is independent of L, the beam length or bay width. The lateral stiffness of the frame with an intermediate, realistic stiffness of the beam can be calculated by standard procedures of static structural analysis. The stiffness matrix of the frame is formulated with respect to three DOFs: the lateral displacement u and the rotations of the two beam–column joints. By static condensation or elimination of the rotational DOFs, the lateral force–displacement relation $F = ku$ is determined.

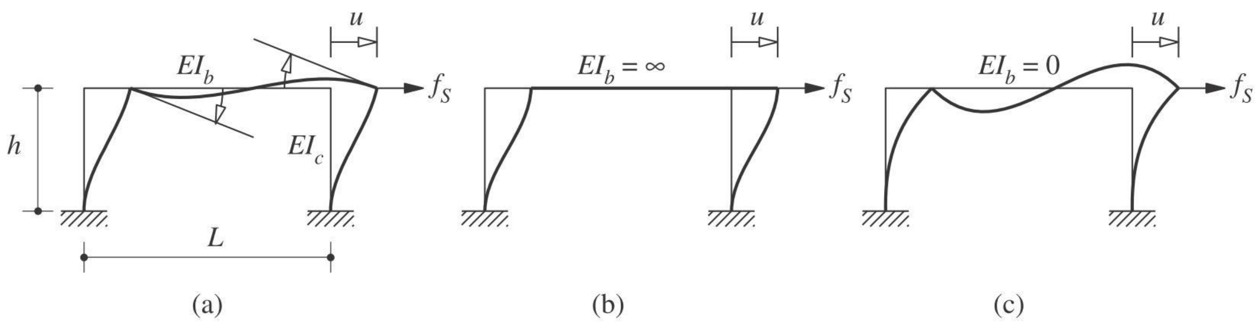


Figure 1-28: A portal frame subjected to lateral force f_s, (b) An extreme case when the beam is rigid, i.e. having infinite stiffness, and (c) Another extreme case when beam has zero stiffness (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

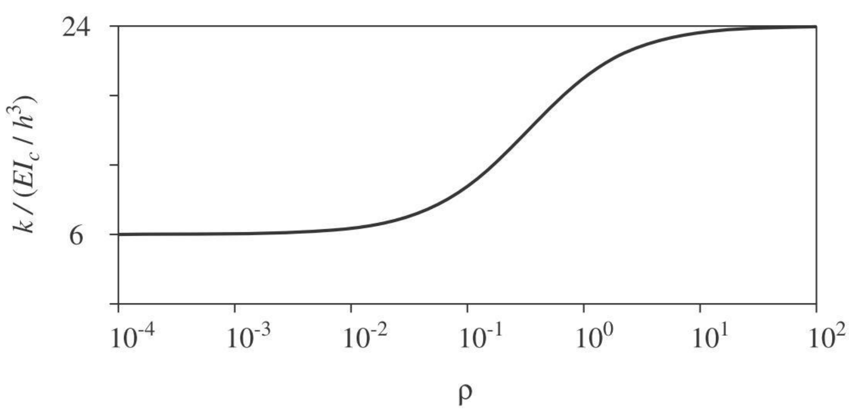


Figure 1-29: Variation of lateral stiffness, k, with beam-to-column stiffness ratio, ρ. (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

If shear deformations in elements are neglected, the result can be written in the form

$$k = \frac{24EI_c}{h^3}\,\frac{12\rho + 1}{12\rho + 4}$$

where $\rho = (EI_b/L) \div (2EI_c/h)$ is the beam-to-column stiffness ratio. The lateral stiffness is plotted as a function of ρ in Figure 1-29; it increases by a factor of 4 as ρ increases from zero to infinity.

1.9.3. Energy Dissipating Mechanism and Damping Force

The process by which vibration steadily diminishes in amplitude is called damping. The kinetic energy and strain energy of the vibrating system are dissipated by various damping mechanisms that we shall mention later. For the moment, we simply recognize that an energy-dissipating mechanism should be included in the structural idealization in order to incorporate the feature of decaying motion observed during free vibration tests of a structure. The most commonly used damping element is the viscous damper, in part because it is the simplest to deal with mathematically.

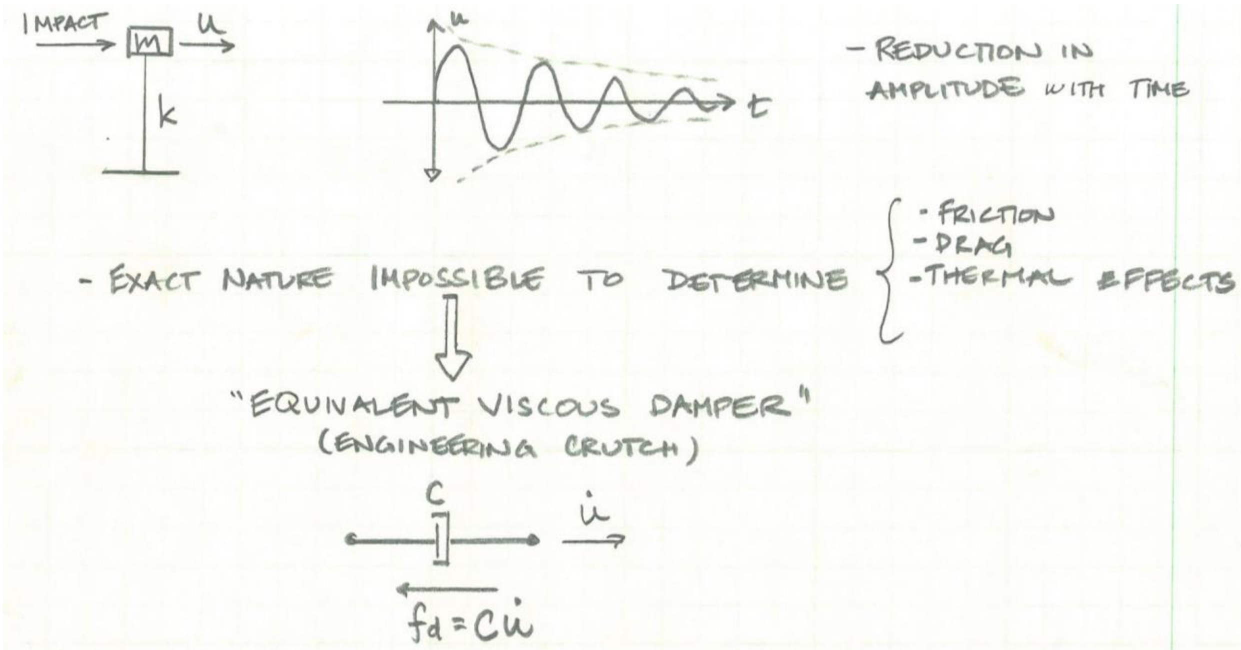


Figure 1-30: Component 3 - The energy dissipating mechanism and damping force

As mentioned earlier, the process by which free vibration steadily diminishes in amplitude is called damping. In damping, the energy of the vibrating system is dissipated by various mechanisms, and often more than one mechanism may be present at the same time. In simple "clean" systems such as the laboratory models of Figure 1-31, most of the energy dissipation presumably arises from the thermal effect of repeated elastic straining of the material and from the internal friction when a solid is deformed. In actual structures, however, many other mechanisms also contribute to the energy dissipation. In a vibrating building these include friction at steel connections, opening and closing of micro-cracks in concrete, and friction between the structure itself and nonstructural elements such as partition walls. It seems impossible to identify or describe mathematically each of these energy-dissipating mechanisms in an actual building.

As a result, the damping in actual structures is usually represented in a highly idealized manner. For many purposes the actual damping in a SDF structure can be idealized satisfactorily by a linear viscous damper or dashpot. The damping coefficient is selected so that the vibrational energy it dissipates is equivalent to the energy dissipated in all the damping mechanisms, combined, present in the actual structure. This idealization is therefore called equivalent viscous damping.

Figure 1-31 (a) shows a linear viscous damper subjected to a force f_D along the DOF u. The internal force in the damper is equal and opposite to the external force f_D (Figure 1-31 (b)). The damping force f_D is related to the velocity $\dot{u}$ across the linear viscous damper by

$$f_D = c\dot{u}$$

where the constant c is the viscous damping coefficient; it has units of force $\times$ time/length. Unlike the stiffness of a structure, the damping coefficient cannot be calculated from the dimensions of the structure and the sizes of the structural elements. This should not be surprising because, as we noted earlier, it is not feasible to identify all the mechanisms that dissipate vibrational energy of actual structures. Thus vibration experiments on actual structures provide the data for evaluating the damping coefficient. These may be free vibration experiments that lead to data. For example, the measured rate at which motion decays in free vibration will provide a basis for evaluating the damping coefficient. The damping property may also be determined from forced vibration experiments.

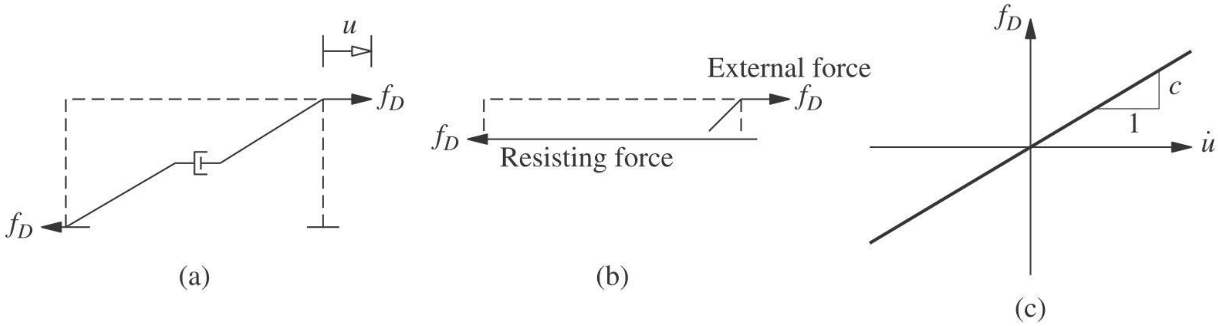


Figure 1-31: (a) The assumed damping mechanism in a portal frame, (b) The damping force, and (c) The linear relationship between f_D and velocity, $\dot{u}$ (Taken from Chopra (2012) Dynamics of Structures, 4th Edition)

The equivalent viscous damper is intended to model the energy dissipation at deformation amplitudes within the linear elastic limit of the overall structure. Over this range of deformations, the damping coefficient c determined from experiments may vary with the deformation amplitude. This nonlinearity of the damping property is usually not considered explicitly in dynamic analyses. It may be handled indirectly by selecting a value for the damping coefficient that is appropriate for the expected deformation amplitude, usually taken as the deformation associated with the linearly elastic limit of the structure.

Additional energy is dissipated due to inelastic behavior of the structure at larger deformations. Under cyclic forces or deformations, this behavior implies formation of a force–deformation hysteresis loop (similar to Figure 1-8). The damping energy dissipated during one deformation cycle between deformation limits ± u_o (Figure 1-21) is given by the area within the hysteresis loop abcda (Figure 1-27 (c)). This energy dissipation is usually not modeled by a viscous damper, especially if the excitation is earthquake ground motion. Instead, the most common, direct, and accurate approach to account for the energy dissipation through inelastic behavior is to recognize the inelastic relationship between resisting force and deformation (such as shown in Figure 1-27 (c)), in solving the equation of motion. Such force–deformation relationships are obtained from experiments on structures or structural components at slow rates of deformation, thus excluding any energy dissipation arising from rate-dependent effects. The usual approach is to model this damping in the inelastic range of deformations by the same viscous damper that was defined earlier for smaller deformations within the linearly elastic range.

Chapter 2

Dynamics of Single-Degree-of-Freedom (SDF) Systems

The essential physical properties of any linearly elastic structural or mechanical system subjected to an external source of excitation or dynamic loading are its mass, elastic properties (flexibility or stiffness), and energy-loss mechanism or damping. In the simplest model of a single-degree-of-freedom (SDF) system, each of these properties is assumed to be concentrated in a single physical element. Figure 2-1 shows some examples of structures that may be represented for dynamic analysis as single-degree-of-freedom (SDF) systems, that is, structures modeled as systems with a single displacement coordinate.

Figure 2-1: Some examples of structures which can be idealized as a single-degree-of-freedom (SDF) systems

These single-degree-of-freedom systems may be described conveniently by the mathematical model shown in Figure 2-2 which has the following elements: (1) a mass element m representing the mass and inertial characteristic of the structure; the entire mass m of this system is included in the rigid block which is constrained by rollers so that it can move only in simple translation; thus, the single displacement coordinate $u(t)$ completely defines its position. (2) a spring element k representing the lateral elastic restoring force and potential energy storage of the structure. This element represents the elastic resistance (provided by the weightless spring of stiffness k) of mass to its displacement, (3) a damping element c representing the frictional characteristics and energy losses of the structure; and (4) an excitation force $p(t)$ representing the external applied dynamic force producing the dynamic response of this system.

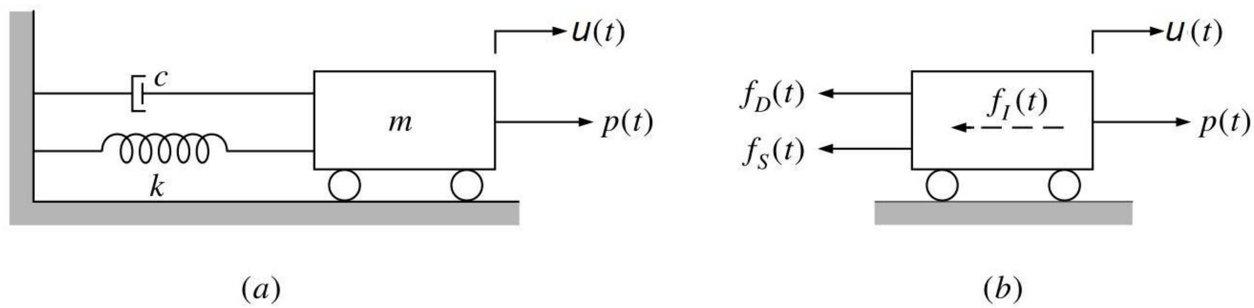


Figure 2-2: An idealized SDF system: (a) basic components; (b) forces in equilibrium (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

Similarly, consider a simple one-story building (idealized as shown in Figure 2-3) subjected to a dynamic force $p(t)$. The entire mass m of this building is assumed to be included in the rigid block which is allowed to move only in simple lateral translation; thus, the single displacement coordinate $u(t)$ completely defines its position as a function of time.

Therefore, at any instantaneous time, the mass m is under the action of four dynamic forces.

1. External dynamic force: $p(t)$
2. Inertia force: $f_I\,(t) = -\frac{md^2u(t)}{dt^2}$
3. Elastic force: $f_s(t) = -ku(t)$, where k is the lateral stiffness of the two columns combined. The negative sign means that the forces is always in the opposite direction to the structural deformation (this is to bring the structure back to its neutral position).
4. Damping force: $f_D(t) = -c\frac{du(t)}{dt} = c\dot{u}(t)$, where c is the damping coefficient of viscous damper. The units of c are force×time/length). The negative sign means that the damping force is always in the opposite direction to velocity du/dt, hence it always dissipates energy.

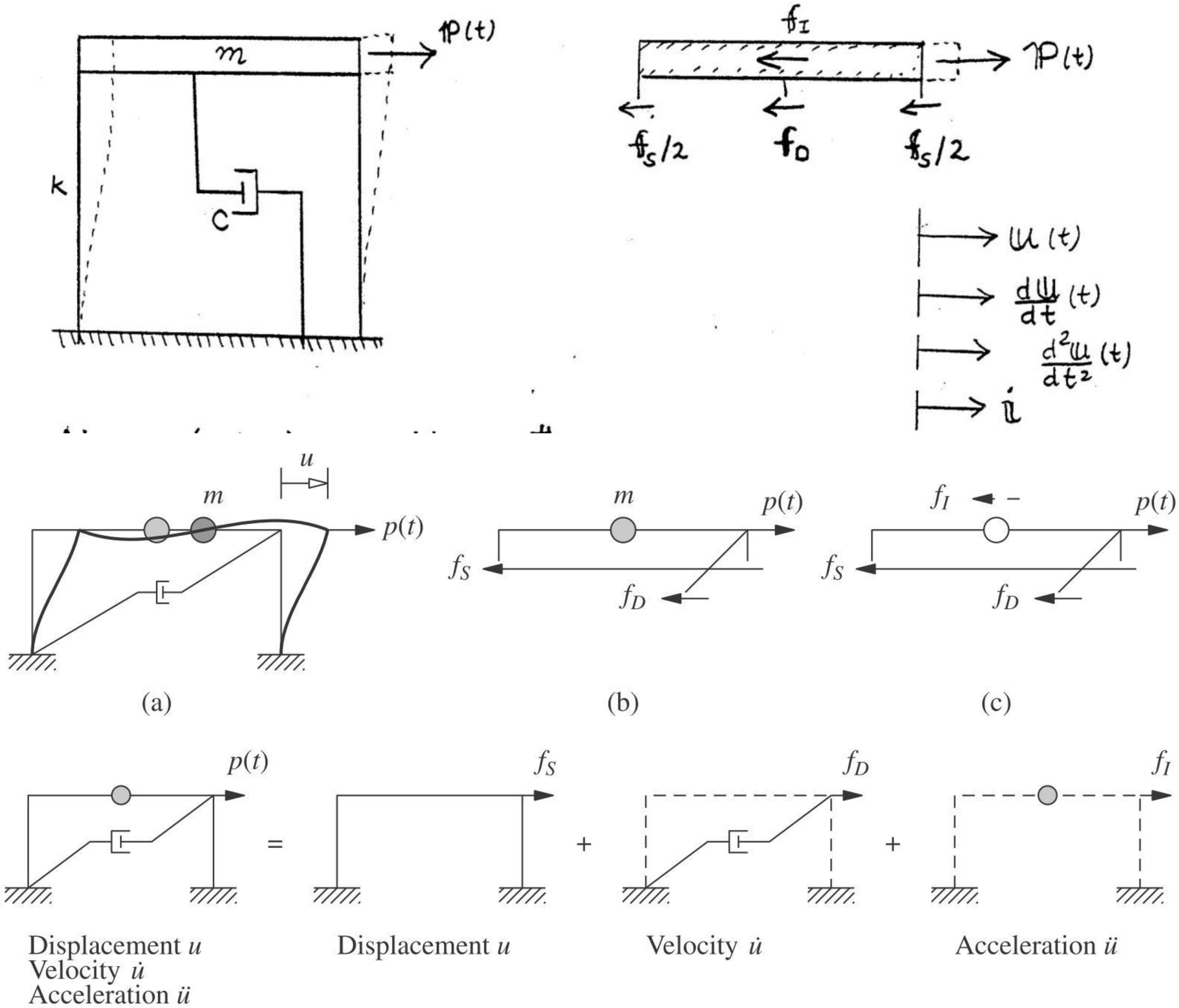


Figure 2-3: (a) An SDF system; (b) stiffness component; (c) damping component; (d) mass component.

2.1. Equation of Motion

The motion of the idealized one-story structure caused by dynamic excitation is governed by an ordinary differential equation, called the "equation of motion". A mentioned in Chapter 1, this equation can be determined using three approaches.

Let's first consider the direct equilibrium approach, i.e. the application of D'Alembert's principle. The sum of all four forces must be zero.

$$\boldsymbol{f_I}(t) + \boldsymbol{f_D}(t) + \boldsymbol{f_s}(t) + \boldsymbol{p}(t) = 0$$

Or

$$m\frac{d^2\boldsymbol{u}(t)}{dt^2} + \frac{cd\boldsymbol{u}(t)}{dt} + k\boldsymbol{u}(t) = p(t)$$

The vector can be converted to scalar function by

$$\boldsymbol{u}(t) = u \, . \, \boldsymbol{i}$$

$$\frac{d\boldsymbol{u}(t)}{dt} = \frac{du}{dt} \, . \, \boldsymbol{i}$$

$$\frac{d^2\boldsymbol{u}(t)}{dt^2} = \frac{d^2u}{dt^2} \, . \, \boldsymbol{i}$$

$$\boldsymbol{p}(t) = p \, . \, \boldsymbol{i}$$

Both p and u are a function of time. Hence, the equation of motion in scalar form is

$$\frac{md^2u(t)}{dt^2} + c\frac{du(t)}{dt} + ku(t) = p(t)$$

This is a second-order linear (ordinary) differential equation. (The equations describing the heat transfer or liquid flow from a porous medium will be first order ordinary differential equations).

The same equation can also be formulated using the principle of virtual work as follows.

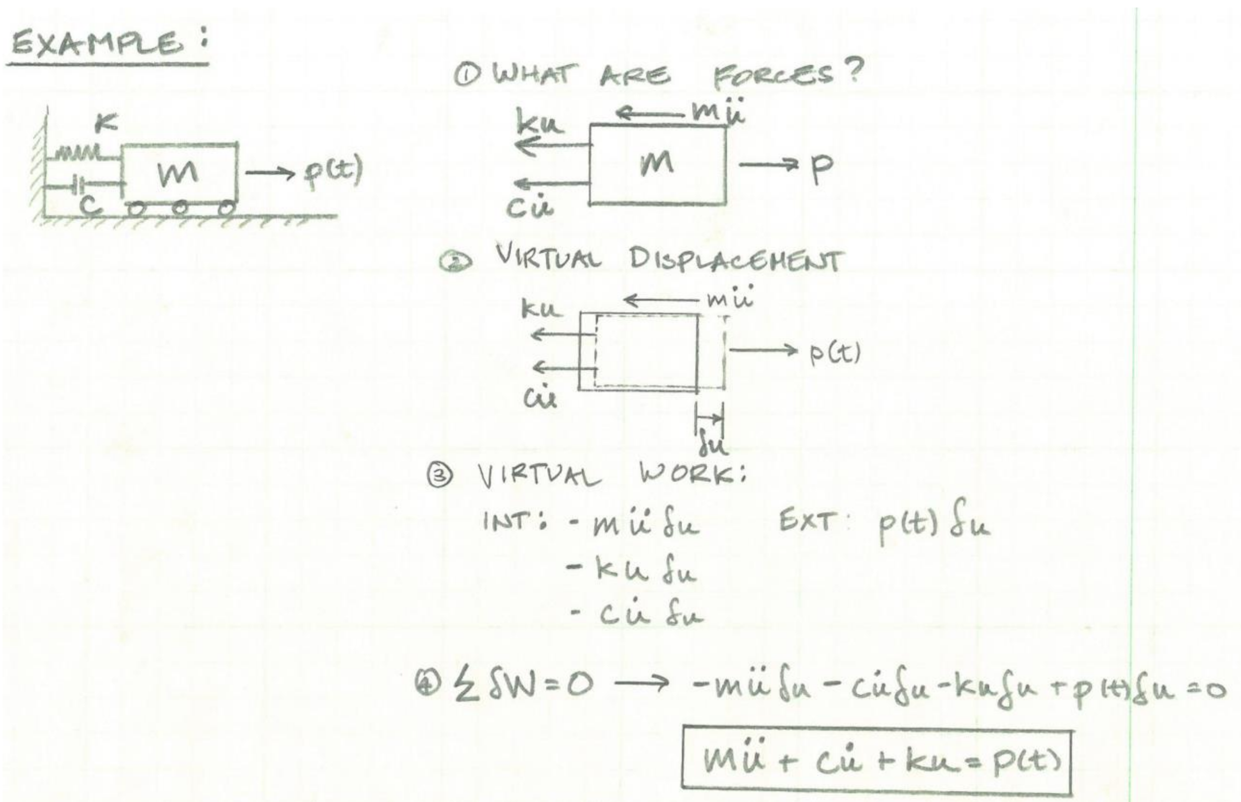


Figure 2-4: The formulation of governing equation of motion for an SDF system using the principle of virtual work

2.1.1. The Basic Knows and Unknowns

In this governing equation of motion, the basic known quantities are the mass of the system (m), applied dynamic load $p(t)$, lateral stiffness of the system (k) and the damping coefficient of the system (c).

The target of structural dynamic is to determine the basic unknown displacement of the system $u(t)$. The other response quantities (e.g. the response velocity du/dt, response acceleration d^2u/dt^2, base shear, overturning moment etc.) can be subsequently derived from $u(t)$.

For example, once the deformation response history $u(t)$ has been evaluated by dynamic analysis of the structure (i.e., by solving the equation of motion), the element forces—bending moments, shears, and axial forces—and stresses needed for structural design can be determined by static analysis of the structure at each instant in time (i.e., no additional dynamic analysis is necessary). This static analysis of a one-story linearly elastic frame can be visualized in two ways:

1. At each instant, the lateral displacement u is known. The joint rotations can be expressed in terms of this lateral displacement and hence can be determined. From the known displacement and rotation of each end of a structural element (beam and column), the element forces (bending moments and shears) can be determined through the element stiffness properties; and stresses can be obtained from element forces.
2. The second approach is to introduce the equivalent static force which is a central concept in earthquake response of structures. At any instant of time t this force f_{eq} is the static (slowly applied) external force that will produce the deformation u determined by dynamic analysis. Thus

$$f_{eq} = ku$$

   where k is the lateral stiffness of the structure. Alternatively, f_{eq} can be interpreted as the external force that will produce the same deformation u in the stiffness component of the structure [i.e., the system without mass or damping] as that determined by dynamic analysis of the structure [i.e., the system with mass, stiffness, and damping]. Element forces or stresses can be determined at each time instant by static analysis of the structure subjected to the force f_{eq} determined from $f_{eq} = ku$.

Note:

The example (idealized one-story) structure in Figure 2-3 is a single-degree-of-freedom system because its motion can be completely describe by only one scalar function – $u(t)$. A 3-story building (as shown below in Figure 2-5) is a three-degree-of-freedom system because at least 3 response functions $(u_1(t), u_2(t), u_3(t))$ are required to completely describe the overall motion of this structure.

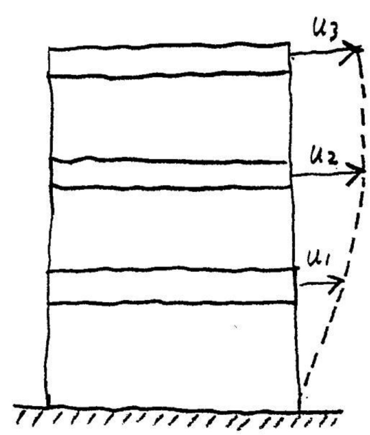


Figure 2-5: A three-degree-of-freedom system

2.1.2. Equation of Motion for an Earthquake

Consider a case when an SDF system is subjected to a ground displacement $\boldsymbol{u}_g(t)$. This represents an earthquake excitation (i.e. a ground motion assumed to be a one-dimensional lateral motion of ground). There is no external force applied to this SDF system.

Let's denote the ground displacement, ground velocity and ground acceleration as $\boldsymbol{u}_g(t), \frac{d\boldsymbol{u_g}(t)}{dt}, \frac{d^2\boldsymbol{u_g}(t)}{dt^2}$.

The total displacement at the roof is defined by $u^t(t)$, where

$$\boldsymbol{u}^t(t) = \boldsymbol{u}_g(t) + \boldsymbol{u}(t)$$

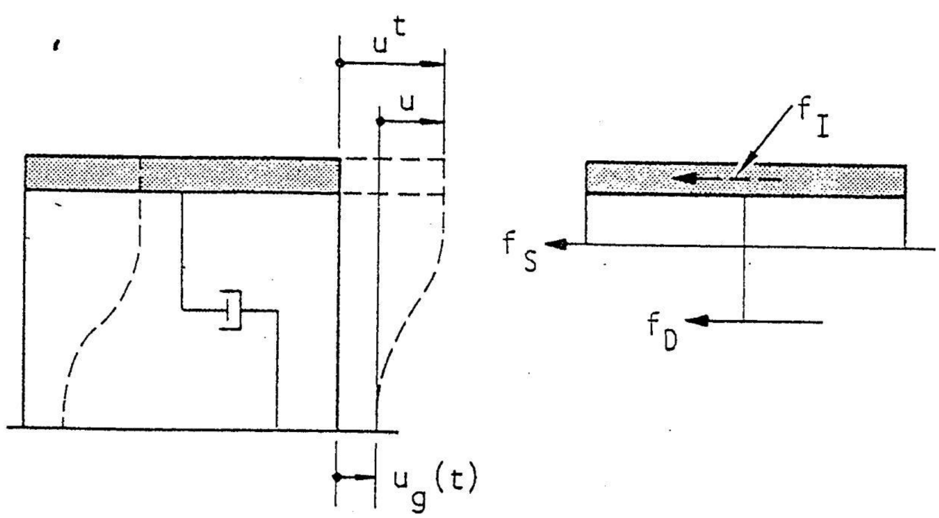
There are three dynamic forces acting on the roof mass:

1. Elastic force $\boldsymbol{f}_S(t) = -ku(t)$
2. Damping force $\boldsymbol{f}_D(t) = -\frac{cdu(t)}{dt}$

Each of these forces is a function of "relative" motion, and not the absolute (or total) motion. However the mass undergoes an acceleration of $\frac{d^2\boldsymbol{ut}}{dt^2}$.

Therefore

3. Inertia force $\boldsymbol{f}_{\mathrm{I}}(\mathrm{t}) = -m\frac{d^2\boldsymbol{u}(t)}{dt^2}(t) = -m\frac{d^2\boldsymbol{u_g}(t)}{dt^2} - m\frac{d^2\boldsymbol{u}(t)}{dt^2}$



(a) Idealized structure (b) Free-body diagram

Figure 2-6: One-story structure subjected to earthquake ground motion $\boldsymbol{u}_g(t)$

Applying the D Alembert's dynamic equilibrium to this case, we get,

$$m\frac{d^2\boldsymbol{u}(t)}{dt^2} + c\frac{d\boldsymbol{u}(t)}{dt} + k\boldsymbol{u}(t) = -m\frac{d^2\boldsymbol{u}_g(t)}{dt^2}$$

In scalar form,

$$m\frac{d^2u(t)}{dt^2} + c\frac{du(t)}{dt} + ku(t) = -m\frac{d^2u_g(t)}{dt^2}$$

This equation of motion is the governing equation of structural deformation $u(t)$, when the structure is subjected to ground acceleration $\frac{d^2u_g(t)}{dt^2}$.

The deformation $u(t)$ of the structure due to ground acceleration $\ddot{u}_g(t)$ is identical to the deformation $u(t)$ of the structure if its base were stationary and if it were subjected to an external force $P_{eff}(t) = -m\ddot{u}_g(t)$.

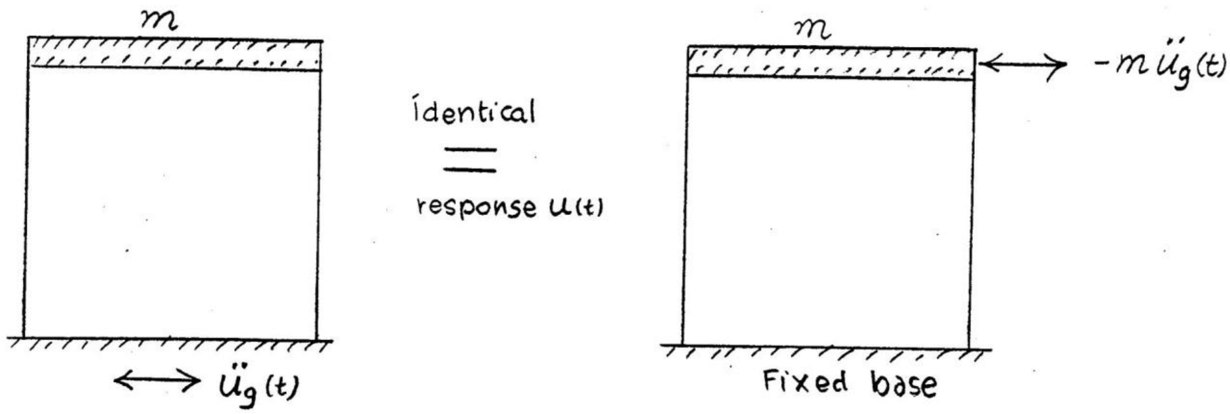


Figure 2-7: One-story building subjected to an earthquake

The negative sign in this effective load definition indicates that the effective force opposes the sense of the ground acceleration. In practice, this has little significance as the engineer is usually only interested in the maximum absolute value of $u(t)$; in this case, the minus sign can be removed from the effective loading term.

If we want to compare how large or damaging an earthquake is, we should compare and check $\ddot{u}_g(t)$ and not $u(t)$.

For structural design against earthquakes, both the total (or absolute) and the relative values of these quantities may be needed. The relative displacement $u(t)$ associated with deformations of the structure is the most important since the internal forces in the structure are directly related to $u(t)$.

2.2. Solution Methods for the Equations of Motion

The equation of motion derived earlier (for a linear SDF system subjected to external force) is the second-order differential equation. For any given excitation, this equation can be solved to determine all responses of a system. The initial displacement $u(0)$ and initial velocity $\dot{u}(0)$ at time zero must also be specified to define the problem completely. Typically, the structure is at rest before the onset of any dynamic excitation, so that the initial velocity and displacement are zero.

The equation of motion derived earlier can be solved using the following four approaches.

2.2.1. Classical Solution

The complete solution of the linear differential equation of motion consists of the sum of the complementary solution $u_c(t)$ and the particular solution $u_p(t)$, that is, $u(t) = u_c(t) + u_p(t)$. Since the differential equation is of second order, two constants of integration are involved. They appear in the complementary solution and are evaluated from the knowledge of the initial conditions.

2.2.2. Duhamel's Integral

Another well-known approach to the solution of linear differential equations, such as the equation of motion of an SDF system, is based on representing the applied force as a sequence of infinitesimally short impulses. The response of the system to an applied force, $p(t)$, at time t is obtained by adding the responses to all impulses up to that time.

For an undamped SDF system subjected to an applied dynamic loading $p(t)$, the displacement $u(t)$ can be determined using the following integral known as Duhamel's integral. This integral will be derived in a later topic while determining the the response of an SDF system subjected to a general dynamic loading.

$$u(t) = \frac{1}{m\omega n} \int_0^t p(\tau)\ sin[\omega_n(t - \tau)]\, d\tau$$

where $\omega_n = \sqrt{k/m}$. Implicit in this result are "at rest" initial conditions. Duhamel's integral is a special form of the convolution integral found in textbooks on differential equations.

Duhamel's integral provides an alternative method to the classical solution if the applied force $p(t)$ is defined analytically by a simple function that permits analytical evaluation of the integral. For complex excitations that are defined only by numerical values of $p(t)$ at discrete time instants, Duhamel's integral can be evaluated by numerical methods.

2.2.3. Frequency-Domain Method

The Laplace and Fourier transforms provide powerful tools for the solution of linear differential equations, in particular the equation of motion for a linear SDF system. Because the two transform methods are similar in concept, here we mention only the use of Fourier transform, which leads to the frequency-domain method of dynamic analysis.

The Fourier transform $P(\omega)$ of the excitation function $p(t)$ is defined by

$$P(\omega) = \mathcal{F}[p(t)] = \int_{-\infty}^{+\infty} p(t)\, e^{-i\omega t}\, dt$$

The Fourier transform $U(\omega)$ of the solution $u(t)$ of the differential equation is then given by

$$U(\omega) = H(\omega)\, P(\omega)$$

where the complex frequency-response function $H(\omega)$ describes the response of the system to harmonic excitation. Finally, the desired solution $u(t)$ is given by the inverse Fourier transform of $U(\omega)$.

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega)\, P(\omega) e^{-i\omega t}\, d\omega$$

Straightforward integration can be used to evaluate the integral of Fourier transform $P(\omega)$, but contour integration in the complex plane is necessary for to solve the integral for $u(t)$. Closed-form results can be obtained only if $p(t)$ is a simple function, and application of the Fourier transform method was restricted to such $p(t)$ until high-speed computers became available.

The Fourier transform method is now feasible for the dynamic analysis of linear systems to complicated excitations $p(t)$ or $\ddot{u}_g(t)$ that are described numerically. In such situations, the integrals of both of above equations are evaluated numerically by the "discrete Fourier transform method" using the "fast Fourier transform algorithm" developed in 1965.

The frequency-domain method of dynamic analysis is symbolized by above two integral equations. The first gives the amplitudes $P(\omega)$ of all the harmonic components that make up the excitation $p(t)$. The second integral equation can be interpreted as evaluating the harmonic response of the system to each component of the excitation and then superposing the harmonic responses to obtain the final response (t) .

The frequency-domain method, which is an alternative to the time-domain method symbolized by Duhamel's integral, is especially useful and powerful for dynamic analysis of structures interacting with unbounded media. Examples are (1) the earthquake response analysis of a structure where the effects of interaction between the structure and the unbounded underlying soil are significant, and (2) the earthquake response analysis of concrete dams interacting with the water impounded in the reservoir that extends to great distances in the upstream direction.

2.2.4. Numerical Methods

The preceding three dynamic analysis methods are restricted to linear systems and cannot consider the inelastic behavior of structures anticipated during earthquakes if the ground shaking is intense. The only practical approach for such systems involves numerical time-stepping methods, which will be presented as a separate topic in later part. These methods are also useful for evaluating the response of linear and nonlinear systems to excitation (applied force $p(t)$ or ground motion $\ddot{u}_g(t)$) which is too complicated to be defined analytically and is described only numerically.

2.3. Free Vibration Response of an SDF System

Let's consider the motion of an SDF system with the applied force set equal to be zero. The determination of this free vibration response would require the solution of the following homogeneous equation.

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0$$

Where $\ddot{u}(t) = \frac{d^2u(t)}{dt^2}$ and $\dot{u}(t) = \frac{du(t)}{dt}$.

A Quick Review of Basic Concepts:

(a) *Solution form :*

Consider a first-order differential equation

$$\frac{du(t)}{dt} + ku(t) = 0$$

$$\frac{du}{dt} = -ku(t)$$

By separation of variables,

$$\frac{du}{u(t)} = -kdt$$

Integrated both sides

$$lnu = -kt + c$$

Where c is an arbitrary constant.

By applying exponential operation

$$e^{lnu} = u = e^{(-kt+c)} = e^{-kt}\, e^{c} = c_o\, e^{-kt}$$

The solution:

$$u(t) = c_o e^{-kt}$$

where c_0 is an arbitrary constant.

It can be shown that the solution of higher order differential equation are also in this exponential form.

(b) Superposition:

If a solution of a homogeneous linear differential equation is the multiplied by a constant, the resulting function is also a solution.

The sum of two solutions is also a solution

Proof:

Let $\phi_1(t)$ and $\phi_2(t)$ be independent solutions of governing differential equation of an SDF system, such that

$$m\ddot{\phi}_1(t) + c\dot{\phi}_1(t) + k\phi_1(t) = 0$$
$$m\ddot{\phi}_2(t) + c\dot{\phi}_1(t) + k\phi_2(t) = 0$$

Substituting $c_1\phi_1(t)$ info the left-hand side of equation of motion, we get

$$m(c_1\ddot{\phi}_1(t)) + c(c_1\dot{\phi}_1(t)) + k(c_1\phi_1(t)) = 0$$
$$c_1[m\ddot{\phi}_1(t) + c\dot{\phi}_1(t) + k\phi_1(t)] = 0$$

Hence $c_1\phi_1(t)$ is also a solution of equation of motion

In similar manner, by a direct substitution of $c_1\phi_1(t) + c_2\phi_2(t)$ into the first equation, it can be shown that $c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution of equation of motion.

(c) Initial Conditions

Consider $u(t) = c_1\phi_1(t) + c_2\phi_2(t)$ as a general solution of governing equation of motion. Since the constants c_1 and c_2 can have any value, the general solution can represent ∞ different solutions.

Usually initial conditions are known and we are seeking for one specific solution that satisfies those initial conditions.

Example of initial conditions:

$u(0)$ and $\dot{u}(0)$ are the initial displacement and initial velocity of the SDF system. Two conditions are needed because there are two unknown arbitrary constants to be specified.

$$u(0) = c_1\phi_1(0) + c_2\phi_2(0)$$
$$\dot{u}(0) = c_1\dot{\phi}_1(0) + c_2\dot{\phi}_2(0)$$

$\phi_1(0), \phi_2(0), \dot{\phi}_1(0), \dot{\phi}_2(0), u(0)$ and $\dot{u}(0)$ all are known. Therefore c_1 and c_2 can be determined.

For more details, see Erwin Kreyszig's Advanced Engineering Mathematics, John Wiley & Sons.

Now consider the equation governing the free vibration of an SDF system as follows.

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0$$

Assume the solution in the exponential form:

$$u(t) = Ge^{st}$$

Where G and s are constants. Substituting this solution back into the equation of motion,

$$ms^2Ge^{st} + csGe^{st} + kGe^{st} = 0$$

$$(ms^2 + cs + k)Ge^{st} = 0$$

To have a non-zero solution of $u(t)$, the $ms^2 + cs + k$ must be zero,

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

2.3.1. Undamped Free vibration Response

In this case, $c = 0$.

Introducing the notation

$$\omega = \sqrt{k/m}$$

The above equation becomes,

$$s^2 + \omega^2 = 0$$

Which has two solutions,

$$s = \pm i\omega$$

Where $i = \sqrt{-1}$

Hence the general solution of $u(t)$ is

$$u(t) = G_1e^{i\omega t} + G_2e^{-i\omega t}$$

Where G_1 and G_2 are arbitrary constants.

Since there are two arbitrary constants, two initial conditions need to specified, i.e. $u(0)$ and $\dot{u}(0)$.

$$u(0) = G_1e^0 + G_2e^0 = G_1 + G_2$$

$$\dot{u}(0) = i\omega G_1e^0 - i\omega G_2e^0 = i\omega G_1 - i\omega G_2$$

Therefore,

$$G_1 = \frac{1}{2}\left(u(0) + \frac{\dot{u}(0)}{i\omega}\right)$$

$$G_2 = \frac{1}{2}\left(u(0) - \frac{\dot{u}(0)}{i\omega}\right)$$

Note:

Taylor Series:

$-\infty < x < \infty$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{i\omega t} = 1 + i\omega t + \frac{(i\omega t)^2}{2!} + \frac{(i\omega t)^3}{3!} + \cdots$

$e^{i\omega t} = 1 + i\omega t + (-1)\frac{(\omega t)^2}{2!} + (-1)\frac{i(\omega t)^3}{3!} + \cdots$

$e^{i\omega t} = \left\{1 - \frac{(\omega t)^2}{2!} + \cdots\right\} + i\left\{\omega t - \frac{(\omega t)^3}{3!} + \cdots\right\}$

Taylor series of $\cos(\omega t)$ is

$1 - \frac{(\omega t)^2}{2!} + \cdots$

Similarly, the Taylor series of $\sin(\omega t)$ is

$\omega t - \frac{(\omega t)^3}{3!} + \cdots$

Therefore,

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

This is called Euler's equation.

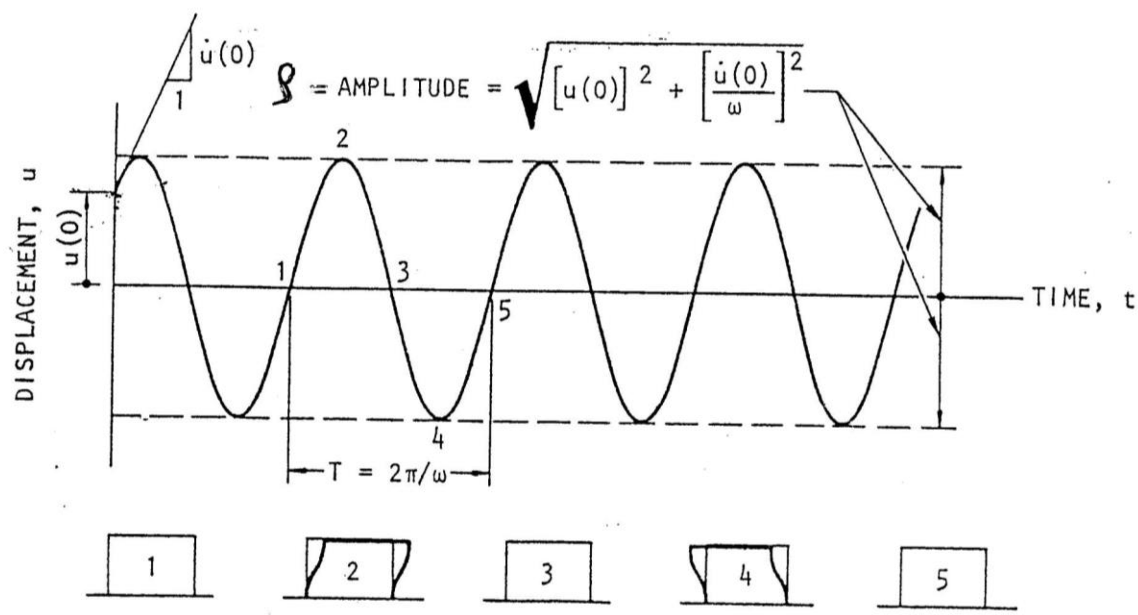
Introducing Euler's equations:

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

Introducing the expressions for G_1 and G_2 into the general solution, we obtain

$$u(t) = u(0)\cos(\omega t) + \frac{\dot{u}(0)}{\omega}\sin(\omega t)$$

It is easy to verify that this equation is the solution of governing equation of motion by direct substitution.



Deformed position of structure corresponding to location 1, 2, 3, 4 and 5 on response-time plot

Figure 2-8: Undamped free vibration of an SDF system

The structure vibrates in simple harmonic motion (or oscillation).

The amplitude of oscillation depends upon $u(0)$ and $\dot{u}(0)$. The above equation may be transformed into

$$u(t) = \rho \cos(\omega t - \theta)$$

Where

$$\rho = \sqrt{\left(u(0)\right)^2 + \left(\frac{\dot{u}(0)}{\omega}\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{\dot{u}(0)}{\omega u(0)}\right)$$

The oscillation does not decay because the structure is undamped. The period of oscillation T is the time required for one cycle of free oscillation. For undamped structure,

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Where ω is the natural circular frequency, f is the natural (cyclic) frequency (cycle/sec, Hz) and T is the natural period (sec). This term "natural" is used to qualify each of the above quantities to emphasize the fact that these are “natural properties” of the structure. These properties are independent of the initial conditions.

The natural (cyclic) frequency f of a one-story building is around 1 Hz. For 15-story building, it is around 1 Hz. For a 60 to 70 story building, it is 0.2 to 0.3 Hz i.e. the building will take 3 to 5 seconds to complete one oscillation depending upon the ratio of mass to its stiffness.

2.3.2. Damped Free Vibration

In this case $c \neq 0$; i.e. damping is present in the structure. The solutions of $s^2 + \frac{c}{m}s + \frac{k}{m} = 0$ for this case are

$$s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2}$$

The characteristics of “s" depends upon the sign of the term $\left\{\left(\frac{c}{2m}\right)^2 - \omega^2\right\}$

The equation will have distinct real roots, if $\left(\frac{c}{2m}\right)^2 - \omega^2 > 0$

The equation will have complex conjugate root, if $\left(\frac{c}{2m}\right)^2 - \omega^2 < 0$

The equation will have real double roots, if $\left(\frac{c}{2m}\right)^2 - \omega^2 = 0$

Case 1: Underdamped System ($c < 2m\omega$, complex conjugate solution of s)

The damping constant c is a measure of the energy dissipated in a cycle of free vibration or in a cycle of forced harmonic vibration. However, the damping ratio—a dimensionless measure of damping—is a property of the system that also depends on its mass and stiffness.

Let's define c_c: critical damping: $c_c \equiv 2m\omega$

Let's define ξ: critical damping ratio; $\xi \equiv c/c_c = c/2m\omega$

Hence, in underdamped systems, $0 < \xi < 1$

Rewriting the solution in terms of ξ, we get

$$s = -\xi\omega \pm \sqrt{(\xi\omega)^2 - \omega^2}$$

$$s = -\xi\omega \pm \sqrt{\omega^2(1 - \xi^2)}\,\sqrt{-1}$$

$$s = -\xi\omega \pm i\omega_D$$

Where

$$\omega_D = \omega\sqrt{1 - \xi^2}$$

$$T_D = \frac{T}{\sqrt{1 - \xi^2}}$$

Then the general solution of $u(t)$ is

$$u(t) = G_1 e^{s_1 t} + G_2 e^{s_2 t} = \left(G_1 e^{-\xi\omega t + i\omega_D t} + G_2 e^{-\xi\omega t - i\omega_D t}\right)$$

$$u(t) = e^{-\xi\omega t}\left(G_1 e^{-i\omega_D t} + G_2 e^{-i\omega_D t}\right)$$

When the initial conditions of $u(0)$ are introduced, the constants G_1 and G_2 can be evaluated, and after using Euler's equations we finally obtain,

$$u(t) = e^{-\xi\omega t}\left[\frac{\dot{u}(0) + u(0)\xi\omega}{\omega_D}\sin(\omega_D t) + u(0)\cos(\omega_D t)\right]$$

The response in above equation can also be presented as

$$u(t) = e^{-\xi\omega t}\rho\cos(\omega_D t - \theta)$$

Where

$$\rho = \sqrt{\left(\frac{(\dot{u}(0) + u(0)\xi\omega)}{\omega_D}\right)^2 + \left(u(0)\right)^2}$$

$$\theta = \tan^{-1}\frac{\dot{u}(0) + u(0)\xi\omega}{\omega_D u(0)}$$

The above equations say that the underdamped system in its free vibration stage will oscillate with circular frequency ω_D and with exponentially decreasing amplitude.

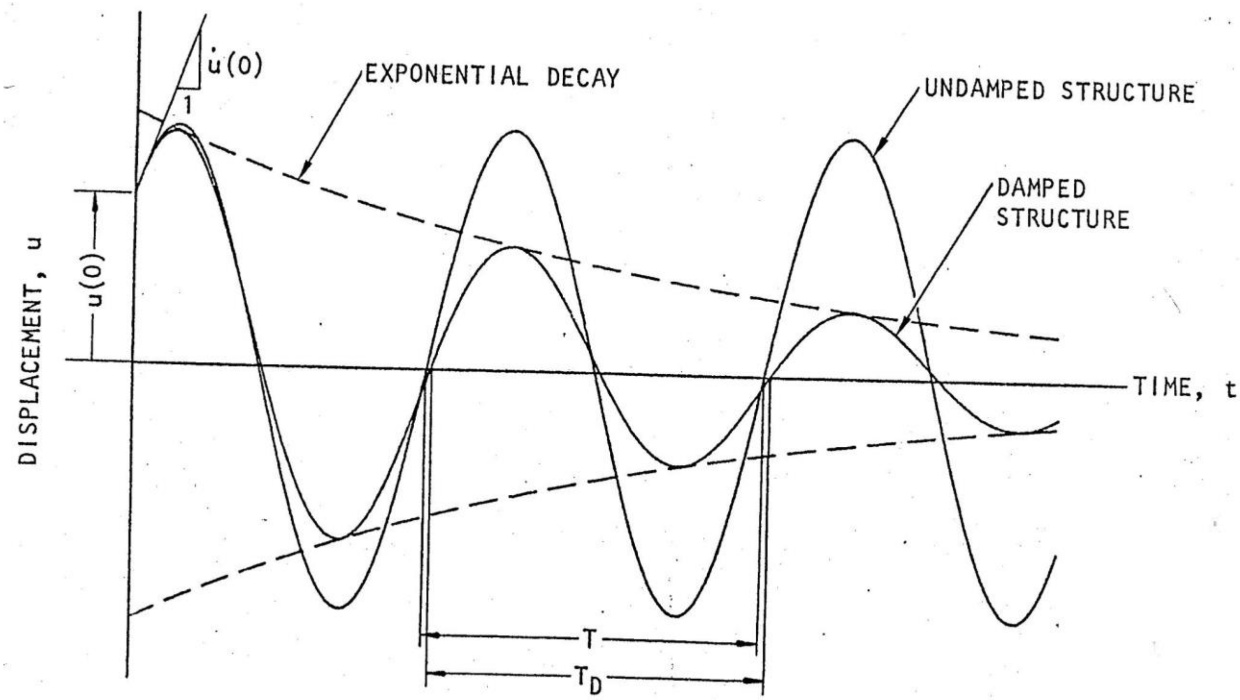


Figure 2-9: The effect of damping on free vibration

In most structures, the critical damping ratio ξ is less than 0.2 (see Figure 2-10 below) and hence, $\omega_D = \omega$ and $T_D = T$. The rate of amplitude decay depends on ξ (see Figure 2-11).

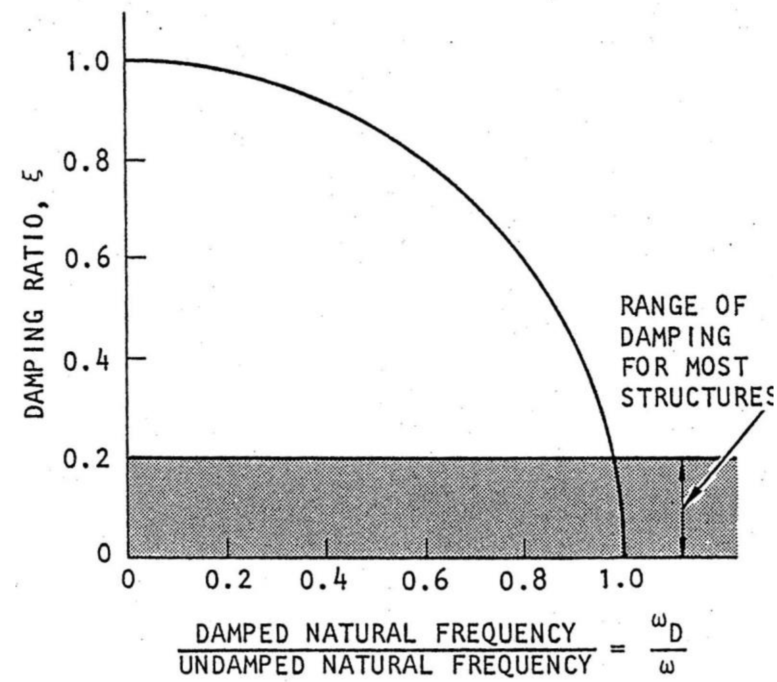


Figure 2-10: The effect of damping on natural frequency of vibration

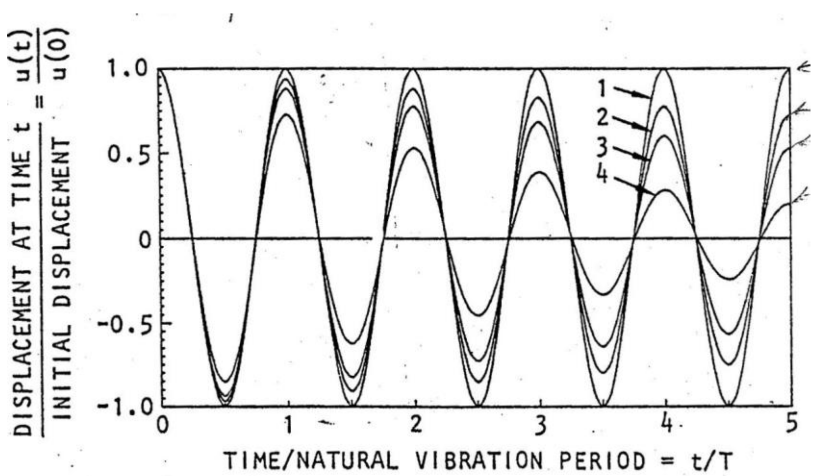


Figure 2-11: The effect of damping on free vibration. Curves 1, 2, 3 and 4 are for damping ratio 0, 1, 2 and 5 percent

In seismic design of most structures, $\xi = 0.05$ is used. For tall buildings subjected to strong winds, we generally assume $\xi = 0.005 - 0.02$. For single cables, $\xi = 0.003 - 0.01$.

Table: Typical damping ratios for common types of construction

Type of Construction	Typical Damping Ratios (ξ)
Steel frame with welded connections and	0.02

flexible walls	
Steel frame with welded connections, normal floors and exterior cladding	0.05
Steel frame with bolted connections, normal floors and exterior cladding	0.1
Concrete frame with flexible internal walls	0.05
Concrete frame with flexible internal walls and exterior cladding	0.07
Concrete frame with concrete or masonry shear walls	0.1
Concrete or masonry shear wall	0.1
Wood frame and shear wall	0.15

Case 2: Critical Damped System ($c = c_c = 2m\omega$)

In this case, $c = c_c = 2m\omega$ and $\xi = 1$. This will yield,

$$s = -\omega$$

The general solution of the governing equation of motion in this case will be of the form.

$$u(t) = G_1 e^{st} + t\, G_2 e^{st} = (G_1 + t\, G_2)e^{-\omega t})$$

The second term must contain t since the two roots of quadratic equation in s are identical.

$$\dot{u}(t) = -\omega(G_1 + t\, G_2)e^{-\omega t}) + G_2 e^{-\omega t}$$

Using initial conditions $u(0)$ and $\dot{u}(0)$, the constants G_1 and G_2 can be determined as follows.

$$G_1 = u(0)$$

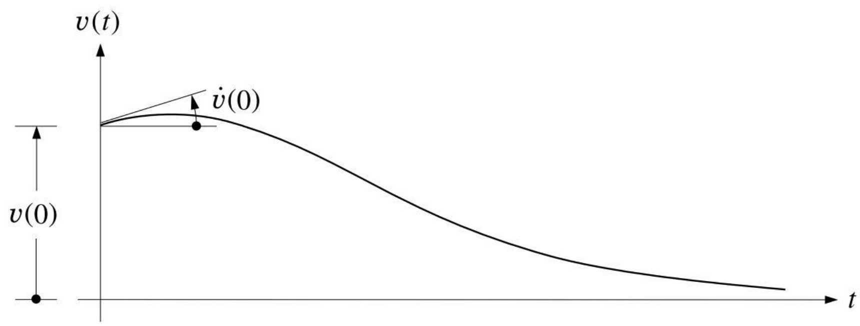
$$G_2 = \dot{u}(0) + \omega\, u(0)$$

The general solution will be,

$$u(t) = [u(0)(1 + \omega t) + \dot{u}(0)t]e^{-\omega t}$$

No oscillations. Damping just eliminated them.

The above expression is shown graphically in Figure below for positive values of $\mathrm{u}(0)$ and $\dot{\mathrm{u}}(0)$. Note that this free response of a critically-damped system does not include oscillation about the zero-deflection position; instead it simply returns to zero asymptotically in accordance with the exponential term of above equation. However, a single zero-displacement crossing would occur if the signs of the initial velocity and displacement were different from each other. A very useful definition of the critically-damped condition described above is that it represents the smallest amount of damping for which no oscillation occurs in the free-vibration response (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).



Free-vibration response with critical damping (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

Case 3: Over Damped System ($c > c_c$)

- Let $\hat{\omega}_n = \omega_n \sqrt{\xi^2 - 1}$

$$u = A_1 e^{(-\xi\omega_n + \hat{\omega}_n)t} + A_2 e^{(-\xi\omega_n - \hat{\omega}_n)t}$$

$$= e^{-\xi\omega_n t}\left(A_1 e^{\hat{\omega}_n t} + A_2 e^{-\hat{\omega}_n t}\right)$$

NOTE:
HYPERBOLIC FUNCT.
$e^{y} = \cosh y - \sinh y$

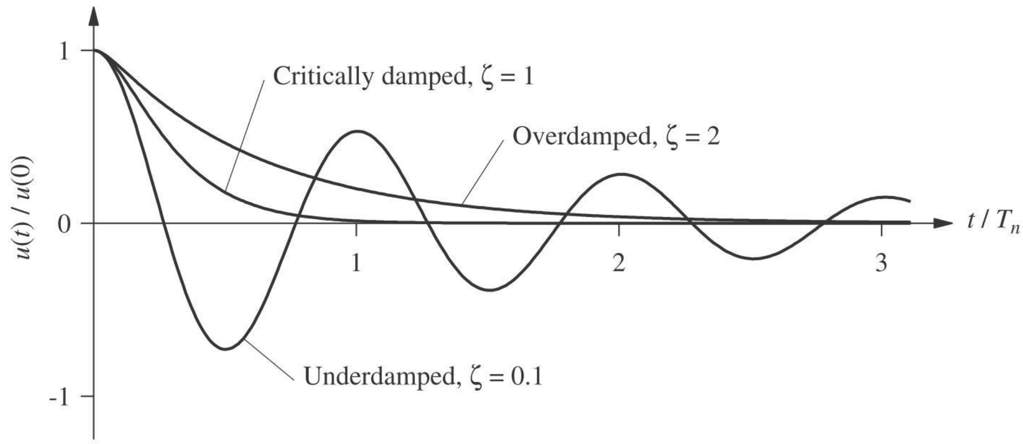
$$= e^{-\xi\omega_n t}\left(A_1 \cosh \hat{\omega}_n t - A_1 \sinh \hat{\omega}_n t + A_2 \cosh \hat{\omega}_n t + A_2 \sinh \hat{\omega}_n t\right)$$

$$= e^{-\xi\omega_n t}\left(C_1 \cosh \hat{\omega}_n t + C_2 \sinh \hat{\omega}_n t\right)$$

- APPLY BOUNDARY CONDITIONS

$$\boxed{u(t) = e^{-\xi\omega_n t}\left(u(0)\cosh \hat{\omega}_n t + \left(\frac{\dot{u}(0) + \xi\omega_n u(0)}{\hat{\omega}_n}\right)\sinh \hat{\omega}_n t\right)}$$

The above equation shows that the response of an overcritically-damped system is similar to the motion of a critically-damped system. However, the asymptotic return to the zero-displacement position is slower depending upon the amount of damping.



Free vibration of under-damped, critically damped, and over-damped systems (Chopra (2012) Dynamics of Structures, 4th Edition)

Figure shows a plot of the motion $u(t)$ due to initial displacement $u(0)$ for three values of ξ. If $c < c_c$ or $\xi < 1$, the system oscillates about its equilibrium position with a progressively decreasing amplitude. If $c = c_c$ or $\xi = 1$, the system returns to its equilibrium position without oscillating. If $c > c_c$ or $\xi > 1$, again the system does not oscillate and returns to its equilibrium position, as in the $\xi = 1$ case, but at a slower rate.

The damping coefficient c_c is called the critical damping coefficient because it is the smallest value of c that inhibits oscillation completely. It represents the dividing line between oscillatory and non-oscillatory motion.

The rest of this presentation is restricted to under-damped systems ($c < c_c$) because structures of interest—buildings, bridges, dams, nuclear power plants, offshore structures, etc.—all fall into this category, as typically, their damping ratio is less than 0.10. There-fore, we have little reason to study the dynamics of critically damped systems ($c = c_c$) or over-damped systems ($c > c_c$) . Such systems do exist, however; for example, recoil mechanisms, such as the common automatic door closer, are overdamped; and instruments used to measure steady-state values, such as a scale measuring dead weight, are usually critically damped. Even for automobile shock absorber systems, however, damping is usually less than half of critical, $\xi < 0.5$. (Chopra (2012) Dynamics of Structures, 4th Edition)

2.3.3. Decay of Free Vibration Response

The true damping characteristics of typical structural systems are very complex and difficult to define. However, it is common practice to express the damping of such real systems in terms of equivalent viscous-damping ratios ξ which show similar decay rates under free-vibration conditions. Therefore, let us now relate more fully the viscous-damping ratio ξ to the free-vibration response (Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

It can be shown that the ratio of any two successive peaks is

$$\frac{u_i}{u_{i+1}} = e^{\left(-2\pi\xi\frac{\omega}{\omega_D}\right)}$$

Taking the natural logarithm on both sides gives the logarithmic decrement δ, as follows.

$$\delta \equiv ln\frac{u_i}{u_{i+1}} = 2\pi\xi\frac{\omega}{\omega_D}$$

Hence for structure with low ξ,

$$\delta \approx 2\pi\xi$$

The above equation is very useful and can be used for the identification of ξ in existing structures. Because it is not possible to determine analytically the damping ratio ξ for practical structures, this elusive property should be determined experimentally. Free vibration experiments provide one means of determining the damping.

Sometimes it is more appropriate to consider the ratio $\frac{u_i}{u_{i+m}}$ where $m > 1$,

$$ln\frac{u_i}{u_{i+m}} = 2m\pi\xi\frac{\omega}{\omega_D}$$

$$\xi \approx \frac{1}{2m\pi}\, ln\left(\frac{u_i}{u_{i+1}}\right)$$

To determine the number of cycles elapsed for a 50% reduction in displacement amplitude, we obtain the following relation from the above equation.

$$m_{50\%} = \frac{0.11}{\xi}$$

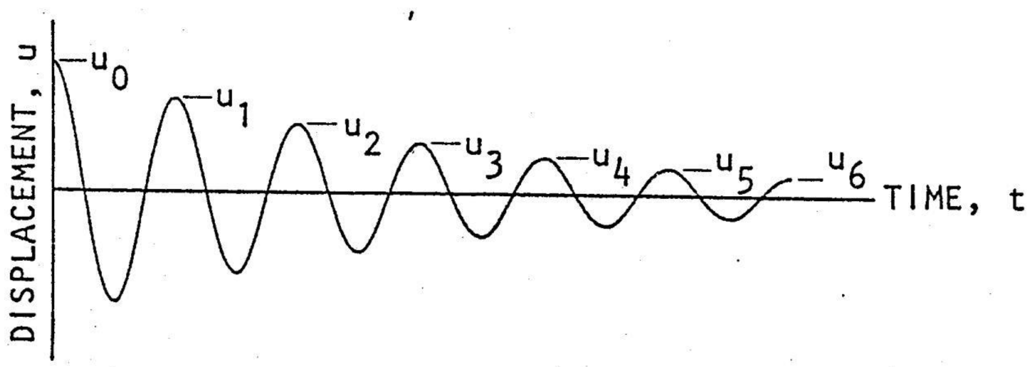
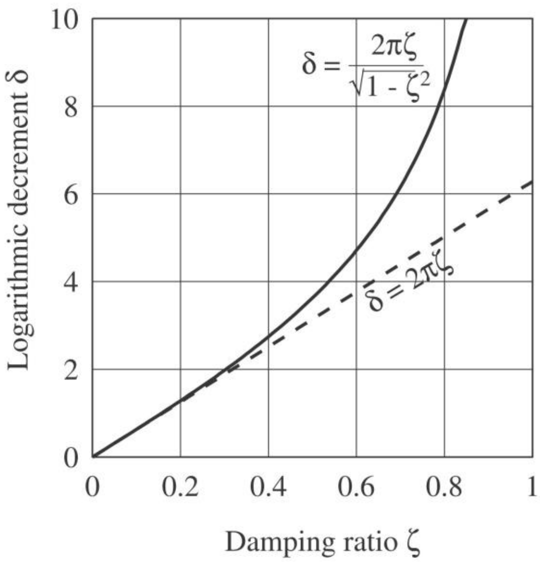
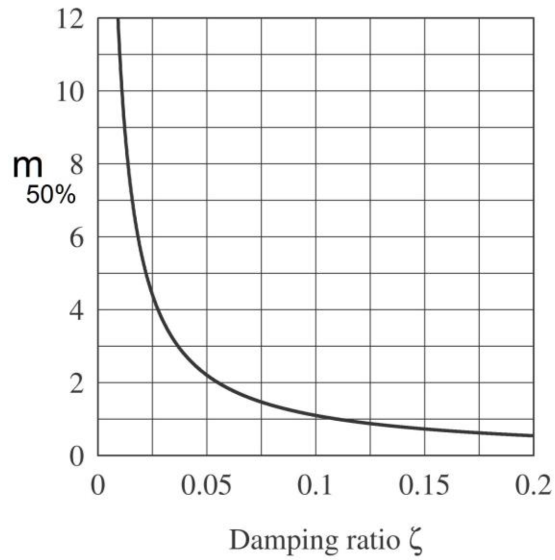


Figure 2-12: Measured displacement response from a free-vibration test

When damped free vibrations are observed experimentally, a convenient method for estimating the damping ratio is to count the number of cycles required to give a 50 percent reduction in amplitude. The relationship to be used in this case is presented graphically below. As a quick rule of thumb, it is convenient to remember that for percentages of critical damping equal to 10, 5, and 2.5, the corresponding amplitudes are reduced by 50 percent in approximately one, two, and four cycles, respectively.



Exact and approximate relations between logarithmic decrement and damping ratio (Chopra (2012) Dynamics of Structures, 4th Edition)

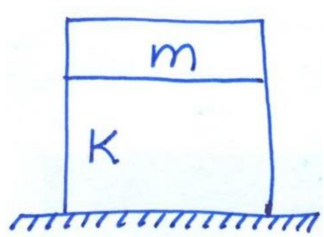


The number of cycles required to reduce the free vibration amplitude by 50% (Chopra (2012) Dynamics of Structures, 4th Edition)

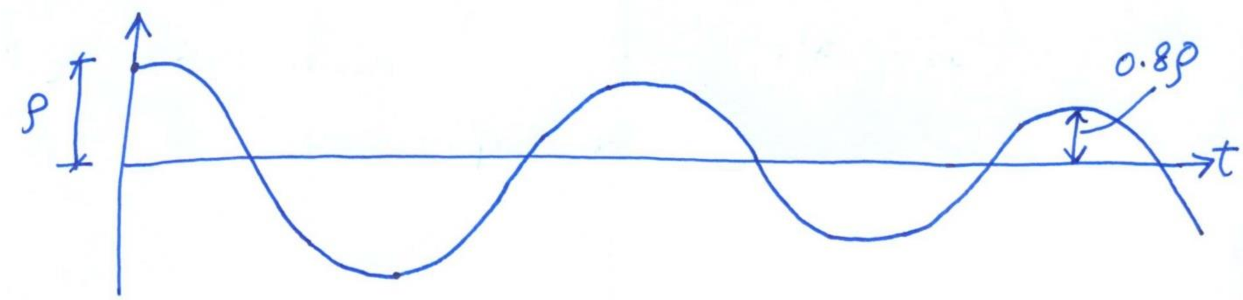
The natural period T_D of the system can also be determined from the free vibration record by measuring the time required to complete one cycle of vibration. Comparing this with the natural period obtained from the calculated stiffness and mass of an idealized system tells us how accurately these properties were calculated and how well the idealization represents the actual structure (Chopra (2012) Dynamics of Structures, 4th Edition).

2.4. Unsolved Examples: Free Vibration Response of an SDF System

Example 1: For an undamped SDF system shown below, f = 3.183 Hz. If 800 kg is added to mass m, the frequency is reduced to f = 2.690 Hz. Determine the mass m and stiffness k.



Example 2: Consider the same SDF system with damping. Suppose that a free vibration response test is applied to this system and the resulting response is shown below.



Determine the damping coefficient c and critical damping ratio ξ.

Example 3: For the same SDF system shown above, determine the free vibration response (both undamped and damped) for the following three initial conditions.

a) $u(0) = 0.03\ m$, $\dot{u}(0) = 0\ m/s$
b) $u(0) = 0.03\ m$, $\dot{u}(0) = 0.2\ m/s$
c) $u(0) = 0.03\ m$, $u(0.5\ sec) = -0.02\ m/s$

For all cases, calculate maximum lateral displacement, velocity and base shear.

Example 4: A simple structure (which can be modeled as a single-degree-of-freedom, SDF system) is shown in Figure Ex.1 below. Its whole mass 10,000 Kg is lumped at top which is supported by two steel columns with hollow cross-sections. The mass is also laterally restrained by a structural component with a given lateral stiffness (represented by a spring in the Figure Ex.1). The columns are firmly fixed to the rigid ground. Important structural dimensions and the column's cross-section are shown in Figure. Modulus of elasticity of steel is $2 \times 10^{11}\ N/m^2$.

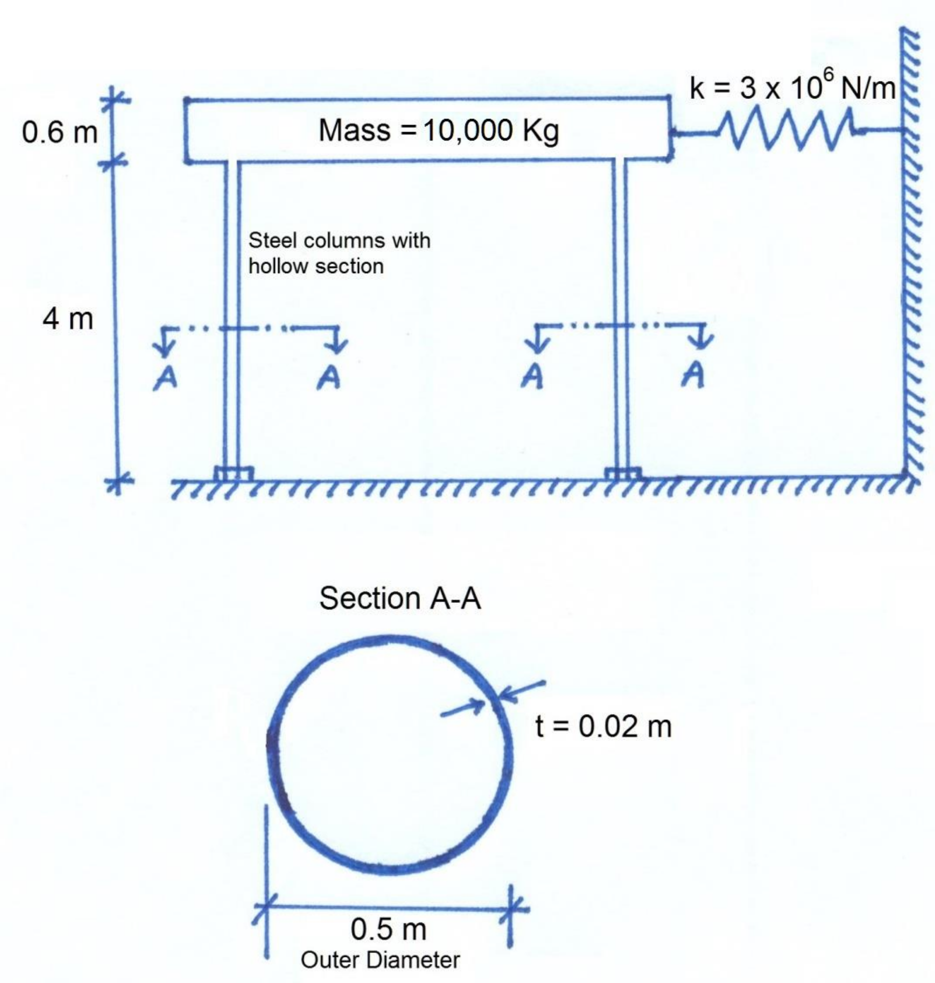


Figure Ex. 1: A simple structure supported by two ciolumns and a lateral spring

a) Determine the natural frequency (f), natural circular frequency (ω) and the natural time period (T) of this simple structure.
b) If the top mass is displaced laterally by 20 mm and released, it will start oscillating (free vibration response). Determine the analytical solution (expression) for the displacement response of this structure and plot it from $t = 0$ to $t = 10$ sec. Assume that the structure doesn't have any energy dissipation.
c) If the critical damping ratio (ξ) of this structure is 0.05, determine the analytical solution (expression) for the damped displacement response of structure and plot it from $t = 0$ to $t = 10$ sec.

Solution:

Lateral stiffness of a column with support conditions shown in Figure Ex. 1 = $3EI/L^3$

The free vibration response of an un-damped SDF system:

$$u(t) = u(0)\cos(\omega t) + \frac{\dot{u}(0)}{\omega} sin(\omega t)$$

Or

$$u(t) = \rho\cos(\omega t - \theta)$$

Where

$$\rho = \sqrt{u^2(0) + \left(\frac{\dot{u}(0)}{\omega}\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{\dot{u}(0)}{\omega u(0)}\right)$$

The free vibration response of a damped SDF system:

$$u(t) = e^{-\xi\omega t}\left[\frac{\dot{u}(0) + u(0)\xi\omega}{\omega_D}\sin(\omega_D t) + u(0)\cos(\omega_D t)\right]$$

Or

$$u(t) = e^{-\xi\omega t}\,\rho\cos(\omega_D t - \theta)$$

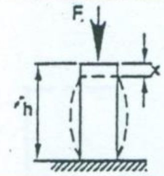
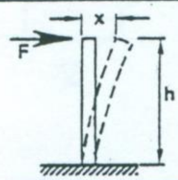
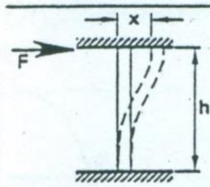
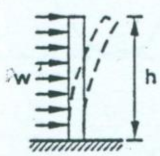
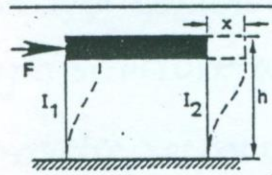
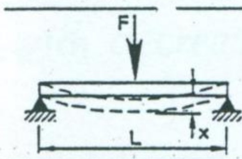
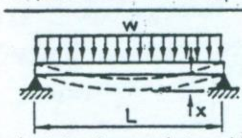
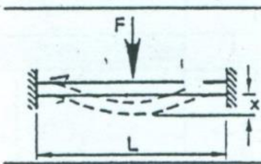
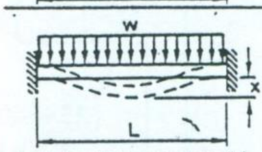
Where

$$\rho = \sqrt{\left(\frac{(\dot{u}(0) + u(0)\xi\omega)}{\omega_D}\right)^2 + \big(u(0)\big)^2}$$

$$\theta = \tan^{-1}\frac{\dot{u}(0) + u(0)\xi\omega}{\omega_D u(0)}$$

Elastic Stiffness

Deflection and stiffness for various systems (due to bending moment only)

System	Maximum Deflection (x)	Stiffness (k)
	$\frac{Fh}{AE}$	$\frac{AE}{h}$
	$\frac{Fh^3}{3EI}$	$\frac{3EI}{h^3}$
	$\frac{Fh^3}{12EI}$	$\frac{12EI}{h^3}$
	$\frac{wL^4}{8EI}$	$\frac{8EI}{L^3}$
	$\frac{Fh^3}{12E(I_1+I_2)}$	$\frac{12E(I_1+I_2)}{h^3}$
	$\frac{FL^3}{48EI}$	$\frac{48EI}{L^3}$
 (w is load per unit length)	$\frac{5wL^4}{384EI}$	$\frac{384EI}{5L^3}$
	$\frac{FL^3}{192EI}$	$\frac{192EI}{L^3}$
 (w is load per unit length)	$\frac{wL^4}{384EI}$	$\frac{384EI}{L^3}$

2.5. Response to Harmonic Loading

In this section, the response of simple structures to harmonic loading is investigated.

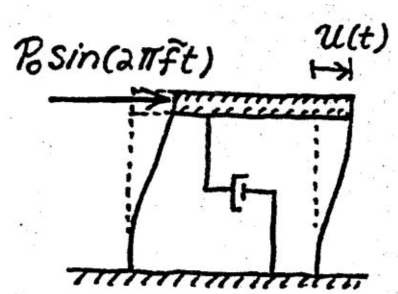


Figure 2-13: A simple structure subjected to a harmonic loading $P(t) = p_o \sin\left(2\pi\bar{f}t\right) = p_o \sin(\bar{\omega}t)$

In mathematic the response is the solution of the following linear nonhomogeneous differential equation:

$$m\ddot{u} + c\dot{u} + ku = p_o \sin(\bar{\omega}t)$$

The solution must also satisfy the prescribed initial conditions: $u(0)$ and $\dot{u}(0)$.

A Quick Review of Basic Concepts:

(a) Solution form :

A general solution $u(t)$ of the linear nonhomogeneous differential equation (as shown above) is the sum of a general solution $u_h(t)$ of the corresponding homogenous differential equation and a particular solution $u_p(t)$.

$$u(t) = u_h(t) + u_p(t)$$

Where

$$m\ddot{u}_h + c\dot{u}_h + ku_h = 0$$

And

$$m\ddot{u}_p + c\dot{u}_p + ku_p = p_o \sin(\bar{\omega}t)$$

$u_p(t)$ is the specific response generated by the form of external force function (In this case eternal force function is harmonic and $u_p(t)$ is also harmonic) and $u_p(t)$ does not need to satisfy the initial conditions.

Introducing the complete general into the governing equation of motion, we obtain

$$m\ddot{\left(u_h(t) + u_p(t)\right)} + c\dot{\left(u_h(t) + u_p(t)\right)} + k\left(u_h(t) + u_p(t)\right)$$
$$= (m\ddot{u}_h + c\dot{u}_h + ku_h) + (m\ddot{u}_p + c\dot{u}_p + ku_p) = 0 + p_o \sin(\bar{\omega}t)$$

2.5.1. Undamped Systems Subjected to Harmonic Loading

$$m\ddot{u} + ku = p_o \sin(\overline{\omega} t)$$

Homogeneous (or Complementary) Solution (Undamped Systems)

From the previous section, we have already obtained $u_h(t)$ as

$$u_h(t) = A\cos(\omega t) + B\sin(\omega t)$$

Particular Solution (Undamped Systems)

The particular solution of a linear second-order differential equation governing the response of an undamped SDF system subjected to harmonic force, is of the form

$$u_p(t) = G\sin(\overline{\omega} t)$$

$$\ddot{u}_p(t) = -G\omega^2 \sin(\overline{\omega} t)$$

Substituting these two values of $\mathrm{u_p(t)}$ and $\mathrm{\ddot{u}_p(t)}$ into the governing equation of motion, we get

$$m\ddot{u}_p + ku_p = p_o \sin(\overline{\omega} t)$$

$$-mG\omega^2 \sin(\overline{\omega} t) + kG\sin(\overline{\omega} t) = p_o \sin(\overline{\omega} t)$$

Solving for G, we get

$$G = \frac{p_0}{k}\frac{1}{1 - (\overline{\omega}/\omega)^2}$$

Therefore,

$$u_p(t) = \frac{p_0}{k}\frac{1}{1 - (\overline{\omega}/\omega)^2}\sin(\overline{\omega} t)$$

The general solution becomes,

$$u(t) = u_h(t) + u_p(t)$$

$$u(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{p_0}{k}\frac{1}{1 - (\overline{\omega}/\omega)^2}\sin(\overline{\omega} t)$$

Now we have to determine A and B,

$$\dot{u}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) + \frac{p_0}{k}\frac{\overline{\omega}}{1 - (\overline{\omega}/\omega)^2}\cos(\overline{\omega} t)$$

This yields,

$$u(0) = A$$

$$\dot{u}(0) = \omega B + \frac{p_0}{k}\frac{\overline{\omega}}{1 - (\overline{\omega}/\omega)^2}$$

Therefore,

$$A = u(0)$$

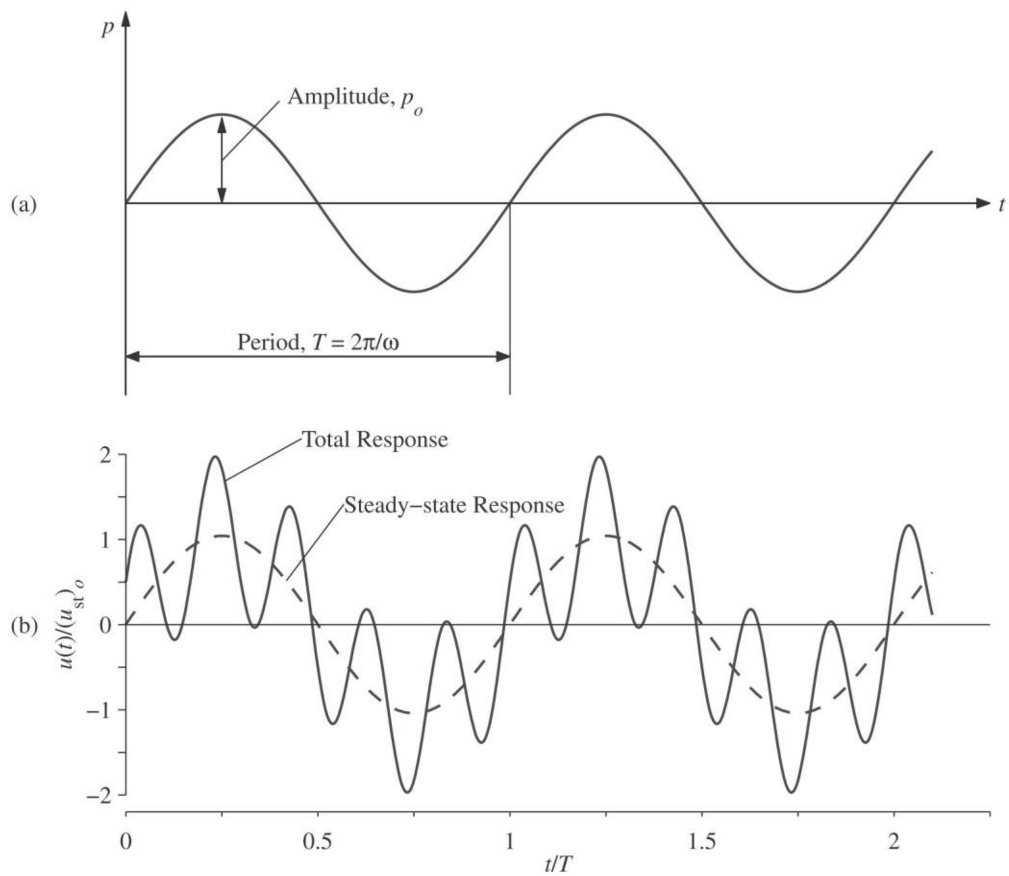
$$B = \frac{\dot{u}(0)}{\omega} - \frac{p_0}{k}\frac{\overline{\omega}/\omega}{1-(\overline{\omega}/\omega)^2}$$

Let's introduce $\beta = \overline{\omega}/\omega$ (frequency ratio).

The general solution becomes,

$$u(t) = u(0)\cos(\omega t) + \left(\frac{\dot{u}(0)}{\omega} - \frac{p_0}{k}\frac{\beta}{1-\beta^2}\right)\sin(\omega t) + \frac{p_0}{k}\frac{1}{1-\beta^2}\sin(\overline{\omega} t)$$

The first two terms show transient vibrations while the third term shows the steady-state response.



(a) Harmonic force; (b) response of undamped system to harmonic force; $\overline{\omega}/\omega = 0.2, u(0) = 0.5p_o/k, \dot{u}(0) = \omega p_o/k$.

$u(t)$ contains two distinct vibration components: (1) the $sin\overline{\omega}t$ term, giving an oscillation at the forcing or exciting frequency; and (2) the $\sin(\omega t)$ and $\cos(\omega t)$ terms, giving an oscillation at the natural frequency of the system. The first of these is the forced vibration or steady-state vibration, for it is present because of the applied force no matter what the initial conditions. The latter is the free vibration or transient vibration, which depends on the initial displacement and velocity. It exists even if $u(0) = u(0) = 0$, in which case the above equation specializes to

$$u(t) = \frac{p_0}{k}\frac{1}{1-\beta^2}\left(\sin(\varpi t) - \beta\sin(\omega t)\right)$$

The transient component is shown as the difference between the solid and dashed lines in above figure where it is seen to continue forever. This is only an academic point because the damping inevitably present in real systems makes the free vibration decay with time. It is for this reason that this component is called transient vibration.

The steady-state dynamic response, a sinusoidal oscillation at the forcing frequency, may be expressed as

$$u(t) = \frac{p_0}{k}\left(\frac{1}{1-\beta^2}\right)\sin(\varpi t) = u_o^{st}\left(\frac{1}{1-\beta^2}\right)\sin(\varpi t)$$

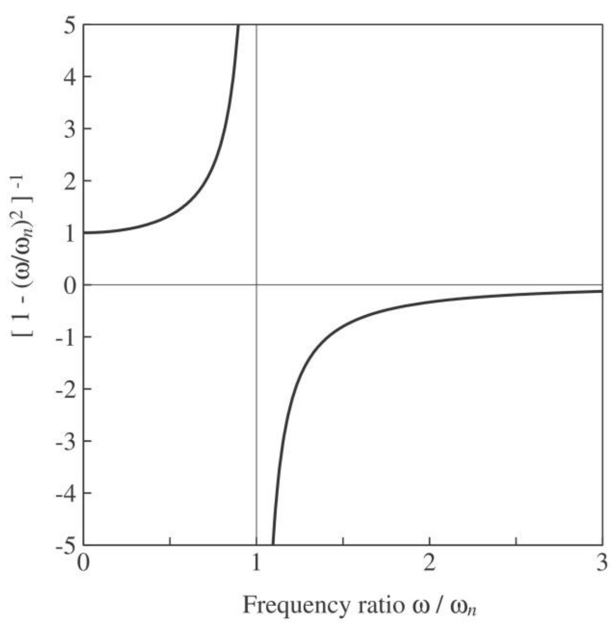
Ignoring the dynamic effects in the governing differential equations of motion of undamped SDF system, the static deformation at each instant is

$$u^{st}(t) = \frac{p_0}{k}\sin(\varpi t)$$

The maximum value of static deformation is

$$u_o^{st} = \frac{p_o}{k}$$

which may be interpreted as the static deformation due to the amplitude po of the force; for brevity we will refer to (u st) o as the static deformation. The factor $\frac{1}{1-(\varpi/\omega)^2}$ or $\frac{1}{1-\beta^2}$ has been plotted in figure below against $\beta = \varpi/\omega$ (the ratio of the forcing frequency to the natural). For $\beta < 1$ or $\varpi < \omega$, this factor is positive, indicating that $u(t)$ and $p(t)$ have the same algebraic sign (i.e., when the force acts to the right, the system would also be displaced to the right). The displacement is said to be in phase with the applied force. For $\beta > 1$ or $\varpi > \omega$, this factor is negative, indicating that $u(t)$ and $p(t)$ have opposing algebraic signs (i.e., when the force acts to the right, the system would be displaced to the left). The displacement is said to be out of phase relative to the applied force.



To describe this notion of phase mathematically, the above equation for steady-state response is rewritten in terms of the amplitude u_o of the vibratory displacement $u(t)$ and phase angle ϕ:

$$u(t) = u_o \sin(\bar{\omega}t - \phi)$$

$$u(t) = u_o^{st}\, R_d \sin(\bar{\omega}t - \phi)$$

Where

$$u_o = u_o^{st}\left(\frac{1}{1-\beta^2}\right)$$

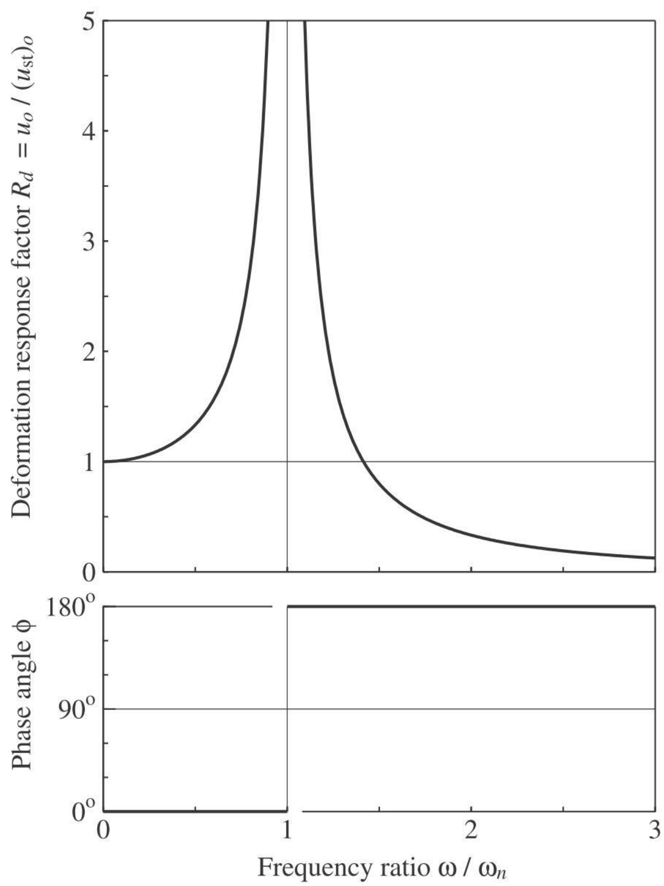
And

$$R_d = \frac{u_o}{u_o^{st}} = \frac{1}{1-\beta^2}$$

$$\phi = 0, \bar{\omega} < \omega$$

$$\phi = 180, \bar{\omega} > \omega$$

For $\bar{\omega} < \omega$, $\phi = 0$, implying that the displacement varies as $\sin\omega t$, in phase with the applied force. For $\bar{\omega} > \omega$, $\phi = 180$, indicating that the displacement varies as $-\sin\omega t$, out of phase relative to the force. This phase angle is shown in figure below as a function of the frequency ratio $\beta = \bar{\omega}/\omega$.



Deformation response factor and phase angle for an undamped system excited by harmonic force.

The deformation (or displacement) response factor R_d is the ratio of the amplitude u_o of the dynamic (or vibratory) deformation to the static deformation (u_o^{st}). The graph between R_d and frequency ratio $\beta = \overline{\omega}/\omega$ permits several observations:

If β is small (i.e., the force is “slowly varying”), R_d is only slightly larger than 1 and the amplitude of the dynamic deformation is essentially the same as the static deformation. If $\beta > \sqrt{2}$ (i.e., $\overline{\omega}$ is higher than $\omega\sqrt{2}$), $R_d < 1$ and the dynamic deformation amplitude is less than the static deformation. As β increases beyond $\sqrt{2}$, R_d becomes smaller and approaches zero as $\overline{\omega}/\omega \rightarrow \infty$, implying that the vibratory deformation due to a “rapidly varying” force is very small. If β is close to 1 (i.e., $\overline{\omega}$ is close to ω), R_d is many times larger than 1, implying that the deformation amplitude is much larger than the static deformation.

The resonant frequency is defined as the forcing frequency at which R_d is maximum. For an undamped system the resonant frequency is ω and R_d is unbounded at this frequency. The vibratory deformation does not become unbounded immediately, however, but gradually, as we demonstrate next.

2.5.2. Resonant Response of Undamped Systems

If $\bar{\omega} = \omega$, the general solution derived above is no longer valid. In this case, the choice of the function $G\sin(\bar{\omega}t)$ for a particular solution fails because it is also a part of the complementary solution. The particular solution now is

$$u_p(t) = G\ t\cos(\omega t)$$

Substituting in governing equation of motion and solving for G, we get

$$G = -\frac{p_o}{2k}\omega$$

and the complete solution is

$$u(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{p_0}{2k}\omega t\cos(\omega t)$$

Now we have to determine A and B,

$$\dot{u}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) - \frac{p_0}{2k}\omega\cos(\omega t) + \frac{p_0}{2k}\omega^2 t\sin(\omega t)$$

This yields,

$$A = u(0)$$

$$B = \frac{\dot{u}(0)}{\omega} + \frac{p_0}{2k}$$

The general solution becomes,

$$u(t) = u(0)\cos(\omega t) + \left(\frac{\dot{u}(0)}{\omega} + \frac{p_0}{2k}\right)\sin(\omega t) - \frac{p_0}{2k}\omega t\cos(\omega t)$$

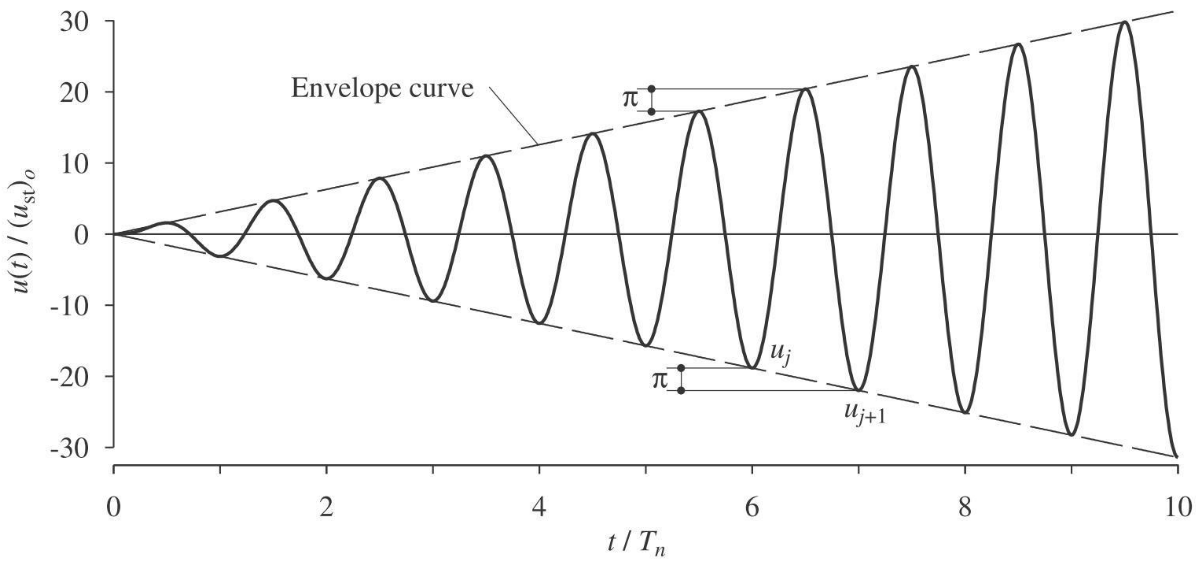
For at-rest initial conditions, $u(0) = u(0) = 0$, the complete solution becomes,

$$u(t) = \frac{p_0}{2k}\big(\sin(\omega t) - \omega t\cos(\omega t)\big)$$

For at-rest initial conditions, $u(0) = u(0) = 0$, the particular solution becomes,

$$u(t) = -\frac{p_0}{2k}\omega t\cos(\omega t)$$

This result is plotted in figure below which shows that the time taken to complete one cycle of vibration is T. The deformation amplitude grows indefinitely, but it becomes infinite only after an infinitely long time.



Response of undamped system to sinusoidal force of frequency $\bar{\omega} = \omega$; $u(0) = u(0) = 0$

This is an academic result and should be interpreted appropriately for real structures. As the deformation continues to increase, at some point in time the system would fail if it is brittle. On the other hand, the system would yield if it is ductile, its stiffness would decrease, and its "natural frequency" would no longer be equal to the forcing frequency, and the general solution derived for resonance case and the above figure would no longer be valid.

2.5.3. Damped Systems Subjected to Harmonic Loading

$$m\ddot{u} + c\dot{u} + ku = p_o \sin(\bar{\omega}t)$$

Homogeneous (or Complementary) Solution (Damped Systems)

From the previous section, we have already obtained $u_h(t)$ as

$$u_h(t) = e^{-\xi\omega t}(A\cos(\omega_D t) + B\sin(\omega_D t))$$

Where A and B are arbitrary constants which satisfy the initial conditions $u(0)$ and $\dot{u}(0)$. The form of equation which we use to determine the general solution was

$$u_h(t) = e^{-\xi\omega t}\left(G_1 e^{i\omega_D t} + G_2 e^{-i\omega_D t}\right)$$

Where G_1 and G_2 were determined such that they satisfy the $u(0)$ and $\dot{u}(0)$.

An alternate form is,

$$u_h(t) = e^{-\xi\omega t}\, \rho_h \cos(\omega_D t - \theta_h)$$

Where ρ_h and θ_h are arbitrary constants determined as a function of initial conditions $u(0)$ and $\dot{u}(0)$.

Particular Solution (Damped Systems)

Particular solution is specific response generated by the form of external force function. Its form depends upon the form of dynamic loading. The specific response to the harmonic force can also be assumed harmonic with a phase lag.

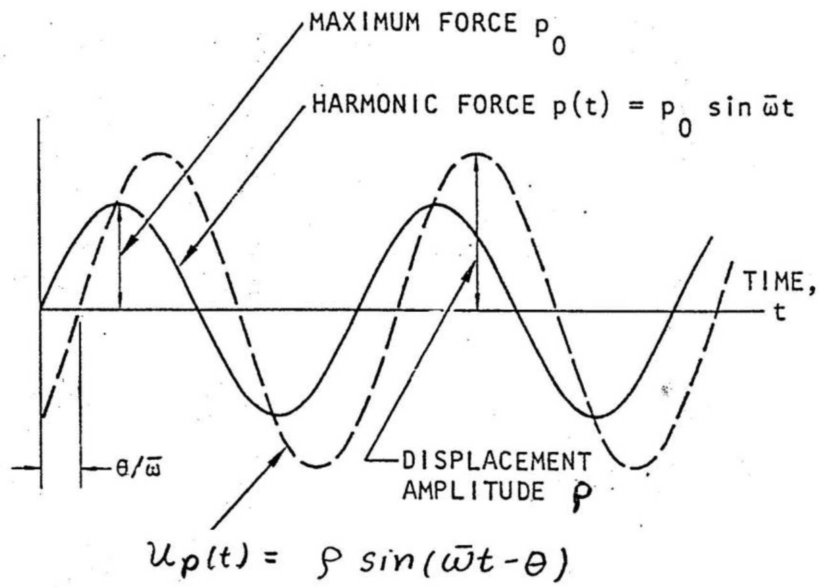


Figure 2-14: The particular solution for harmonic force is also harmonic with a phase log

$$u_p(t) = \rho_p \sin(\bar{\omega} t - \theta_p)$$

In which ρ_p is amplitude and θ_p is phase lag. If $\theta_p = 0$, there will be no lag between applied force $p(t)$ and $u_p(t)$. If $\theta_p = 2\pi$, the response $u_p(t)$will have a lag of one cycle with applied force $p(t)$. For more details about this solution form, please refer to “Advanced Engineering Mathematics” by Erwin Kreyszig).

The particular solution can also be transformed into

$$u_p(t) \ = G'_1 \sin(\bar{\omega} t) + G'_2 \cos(\bar{\omega} t)$$

Where G'_1 and G'_2 are constants to be evaluated.

Employing the previous notations, $\omega^2 = k/m$ and $\xi = c/c_c = c/2m\omega$, we get

$$m\ddot{u}_p + 2\xi m\omega \dot{u}_p + m\omega^2 \ddot{u}_p = p_o \sin(\bar{\omega} t)$$

Substituting the general solution of $u_p(t)$ into above equation and separating the multiples of $\sin(\omega t)$ from the multiples of $\cos(\omega t)$ leads to

$$u_p(t) = G'_1 \sin(\bar{\omega} t) + G'_2 \cos(\bar{\omega} t)$$

$$\dot{u}_p(t) = \bar{\omega} G'_1 \cos(\omega t) - \bar{\omega} G'_2 \sin(\bar{\omega} t)$$

$$\ddot{u}_p(t) = -\bar{\omega}^2 G'_1 \sin(\bar{\omega} t) - \bar{\omega}^2 G'_2 \cos(\bar{\omega} t)$$

$$(-G'_1\overline{\omega}^2 - G'_2\overline{\omega}(2\xi\omega) + \omega^2 G'_1)\sin(\overline{\omega}t) + (-G'_2\overline{\omega}^2 + G'_1\overline{\omega}(2\xi\omega) + \omega^2 G'_2)\cos(\overline{\omega}t) = \frac{p_o}{m}\sin(\overline{\omega}t)$$

Hence,

$$-G'_1\overline{\omega}^2 - G'_2\overline{\omega}(2\xi\omega) + \omega^2 G'_1 = \frac{p_o}{m}$$

$$-G'_2\overline{\omega}^2 - G'_1\overline{\omega}(2\xi\omega) + \omega^2 G'_2 = 0$$

Dividing the above two equations by ω^2 and introducing $\beta = \overline{\omega}/\omega$ (frequency ratio),

$$G'_1(1-\beta^2) - G'_2(2\xi\beta) = \frac{p_o}{k}$$

$$G'_2(1-\beta^2) + G'_1(2\xi\beta) = 0$$

These are two simultaneous algebraic equations for two unknown (G'_1, G'_2). There simultaneous solution yields,

$$G'_1 = \frac{p_o}{k}\frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

$$G'_2 = \frac{p_o}{k}\frac{(-2\xi\beta)}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

Therefore, the particular solution $u_p(t)$ is obtained as

$$u_p(t) = G'_1\sin(\overline{\omega}t) + G'_2\cos(\overline{\omega}t)$$

$$u_p(t) = \frac{p_o}{k}\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}[(1-\beta^2)\sin(\overline{\omega}t) - 2\xi\beta\cos(\overline{\omega}t)]$$

The above equation can also be written as,

$$u_p(t) = \rho_p\sin(\overline{\omega}t - \theta_p)$$

Where,

$$\rho_p = \frac{p_o}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

$$\theta_p = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

General Solution (Damped Systems)

The general solution $u(t)$ is

$$u(t) = u_h(t) + u_p(t)$$

$$u(t) = \left[e^{-\xi\omega t}(A\cos(\omega_D t) + B\sin(\omega_D t))\right] + [G'_1\sin(\overline{\omega}t) + G'_2\cos(\overline{\omega}t)]$$

Where A and B can be determined in terms of initial conditions, similar to undamped case.

The above equation can also be written as follows,

$$u(t) = e^{-\xi\omega t}\,\rho_h \cos(\omega_D t - \theta_h) + \rho_p \sin(\bar{\omega} t - \theta_p)$$

The first term $\left[e^{-\xi\omega t}\,\rho_h \cos(\omega_D t - \theta_h)\right]$ is the free decayed oscillation at ω_D. The ρ_h and θ_h are defined such that $u(0)$ and $\dot{u}(0)$ are satisfied. The oscillations of $u_h(t)$ are quickly damped and eventually become zero if the harmonic force is applied for sufficient time.

The second term $\left[\rho_p \sin\left(\bar{\omega} t - \theta_p\right)\right]$ is constant amplitude oscillation at frequency $\bar{\omega}$ with phase θ_p different from excitation. This term represents the *"steady-state response"*. For most of the real structures, we are mostly interested in this response.

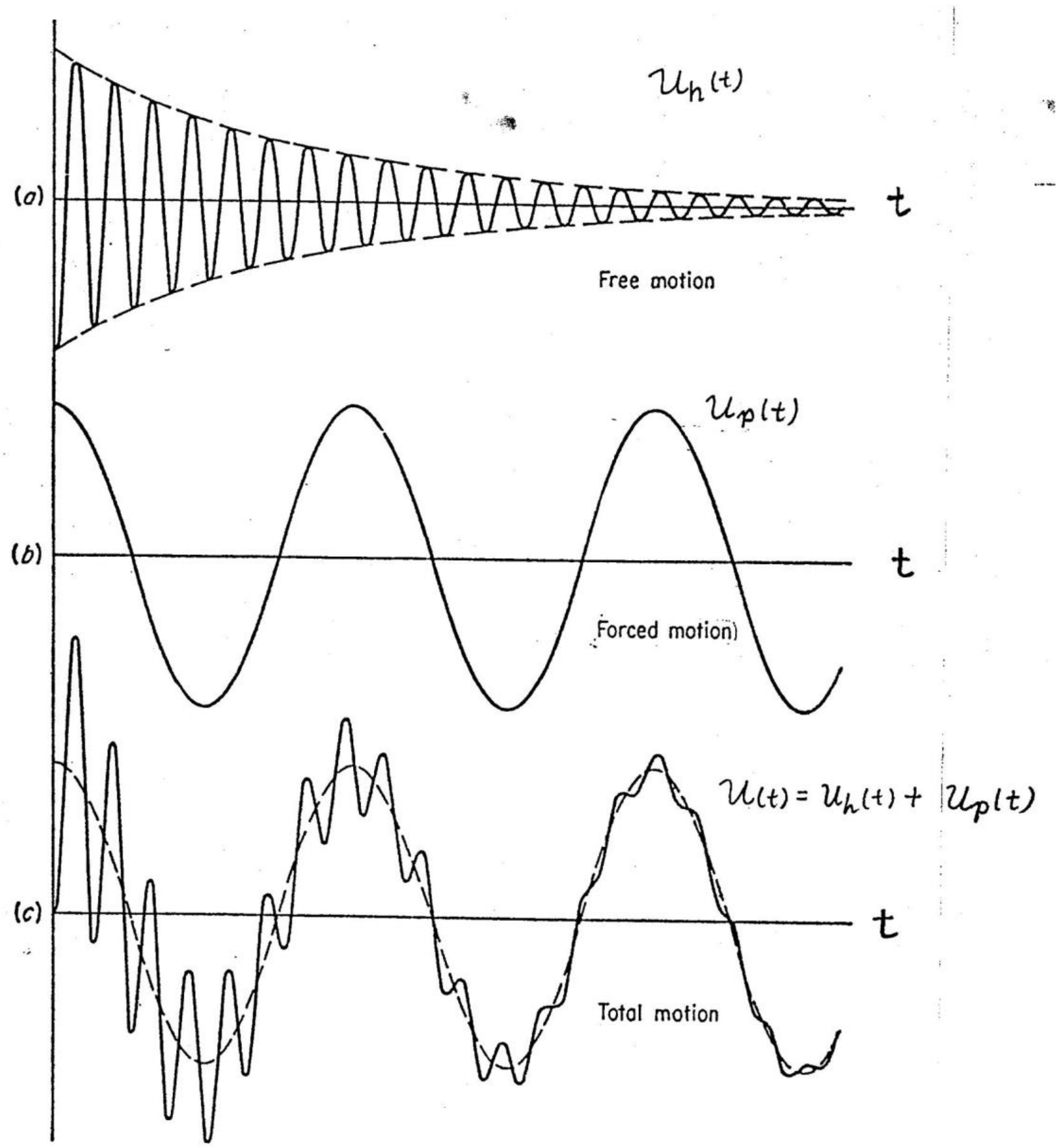


Figure 2-15: The general solution of an SDF system subjected to harmonic excitation

Steady-state Response (Damped Systems)

After sufficient time has passed, $u(t) \rightarrow u_p(t)$. Therefore, $u_p(t)$ is the "steady-state response"

$$u_p(t) = \rho_p \sin(\bar{\omega}t - \theta_p)$$

Where

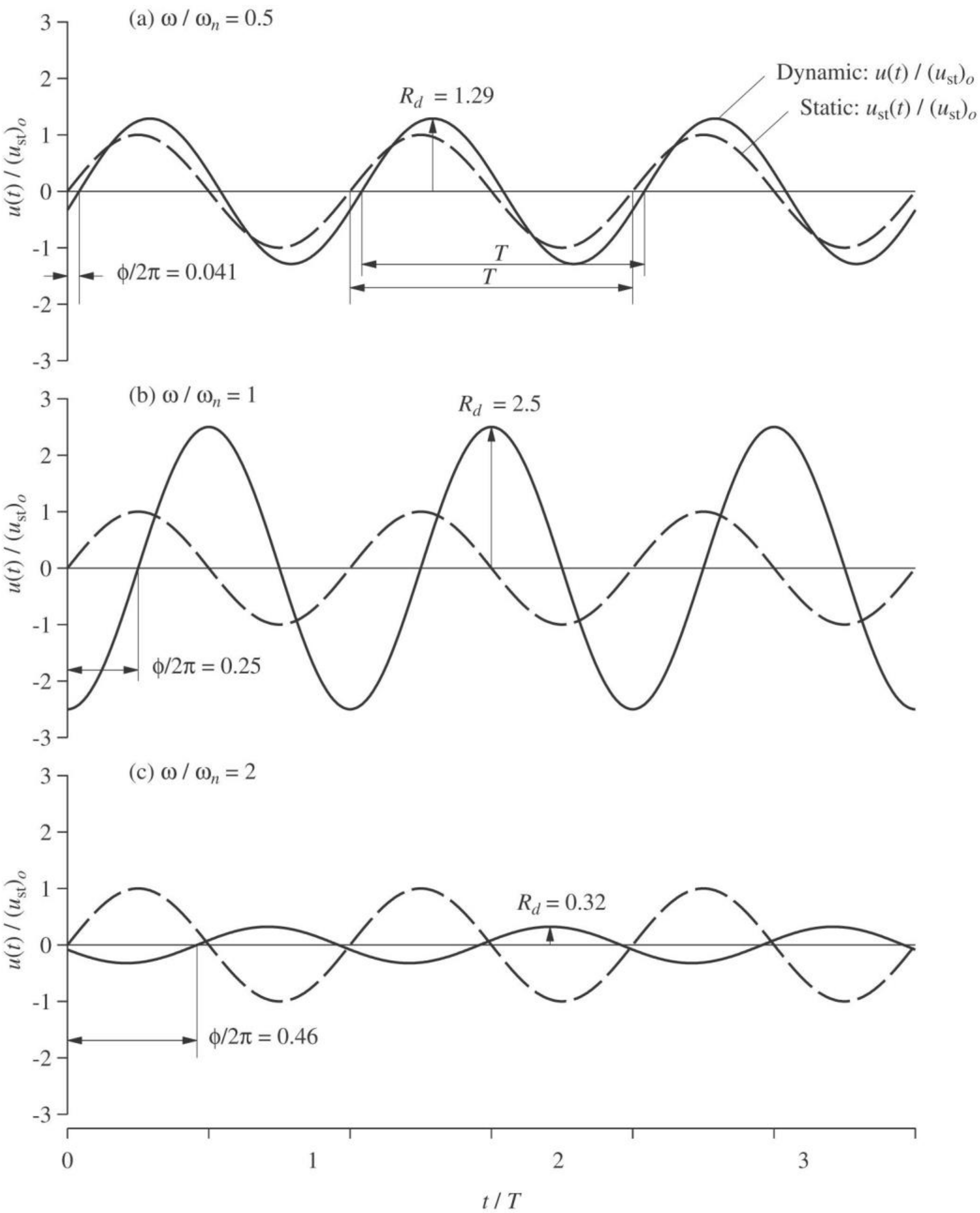
$$\rho_p = \frac{p_o}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} = u_o^{st}\ R_D$$

$$\theta_p = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

$$R_D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

The term p_o/k is the maximum static displacement (u_o^{st}). It is the displacement of structure that would occur if the maximum force p_o were applied as a static force. R_D is a dimensionless factor known as the "*dynamic magnification factor"* or *"displacement response factor"*.

Maximum dynamic displacement (ρ) = maximum static displacement x dynamic magnification factor



Steady-state response of damped systems (ξ= 0.2) to sinusoidal force for three values of the frequency ratio: (a) β= 0.5, (b) β= 1, (c) β= 2.

R_D is a function of

a) Frequency ratio $\beta = \overline{\omega}/\omega$
b) Critical damping ratio ξ

A plot of the amplitude of a response quantity against the excitation frequency is called a frequency-response curve. Figure 2-16 shows the plot of R_D against β for structures with ξ = 0, 0.1. 0.2, 0.5 and 1. Damping reduces R_D and hence the deformation amplitude at all excitation frequencies.

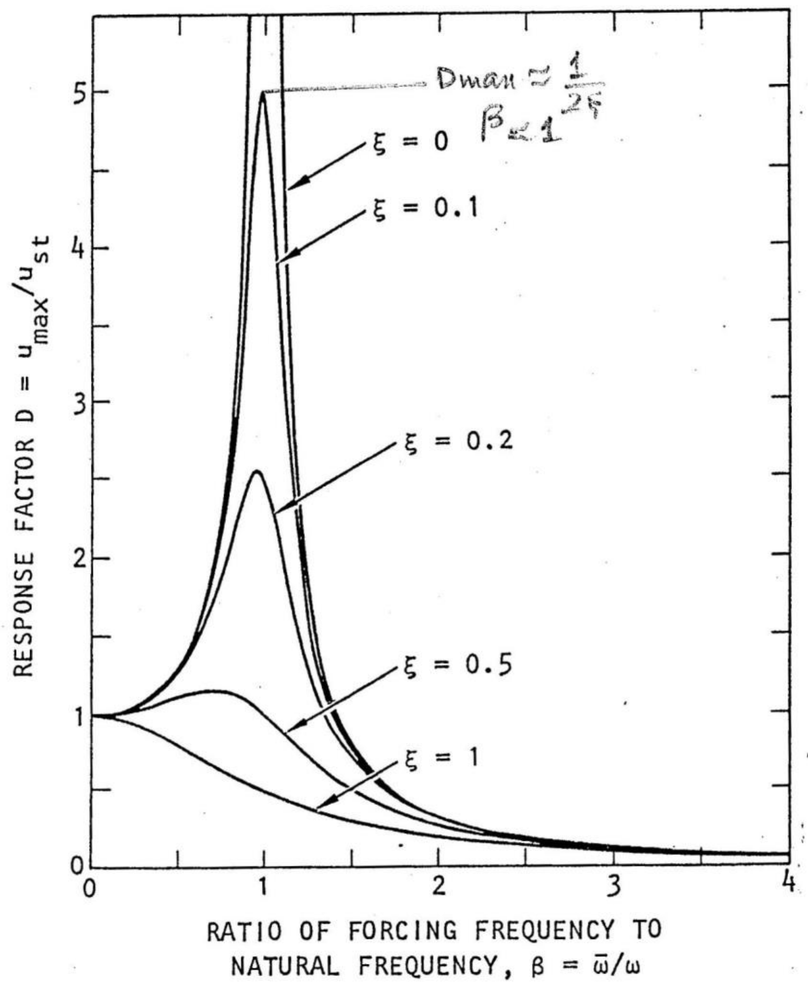


Figure 2-16: The dynamic amplification factor (or response factor) for one-story structure subjected to harmonic force

The magnitude of this reduction is strongly dependent on the excitation frequency. Several discussions can be made as follows:

(1) When β approaches to zero R_D→1, and the dynamic displacement amplitude is about the same as the static one. In the other words, if the forcing frequency $\bar{\omega}$ is much lower than the natural frequency of the structure, the dynamic effects are negligible.
The displacement is controlled by the stiffness of structure, with little effect of mass and damping, so we call this range (β → 0) as “pseudo static” range.

$$u_o \cong u_o^{st} = \frac{p_o}{k}$$

(2) On the other extreme, β >> 1, R_D → 0, $\bar{\omega}$ >> ω. If the forcing frequency ($\bar{\omega}$) is much higher than the natural frequency of the structure (ω), the displacement approaches to zero. In this extreme, the inertia forces dominate. So we call this range “inertial range”.
This result implies that the response is controlled by the mass of the system.

$$u_o \cong u_o^{st}\beta^2 = \frac{p_o}{m\omega^2}$$

(3) Between the two extremes, there is a range where the displacement can be very large when damping ratio is low. This is the range where β is close to 1. At β = 1, R_D → peak, i.e. a small force can produce a very large response.

$$R_D = \frac{1}{2\xi}$$

This result implies that the response is controlled by the damping of the system. Dynamic magnification factor is inversely proportional to damping. In this range, the damping force plays a very crucial role. So, we call this range, “resonant range”.

To give you some ideas about this “resonant amplification”,

ξ of steel structures ≈ $0.01,\ R_D = 1/(2 \times 0.01) = 50$

ξ of concrete structures ≈ $0.05,\ R_D = 1/(2 \times 0.05) = 10$

ξ of tall buildings (300 m to 400 m high), long span bridges (300 m up span) = $0.005,\ R_D = 100$

$$u_o \cong \frac{u_o^{st}}{2\xi} = \frac{p_o}{c\omega}$$

The phase angle ϕ, which defines the time by which the response lags behind the force, varies with $\beta = \bar{\omega}/\omega$ as shown in Figure below. It is examined next for the same three regions of the excitation-frequency scale:

1) If $\beta = \bar{\omega}/\omega \ll 1$ (i.e., the force is “slowly varying”), ϕ is close to $0°$ and the displacement is essentially in phase with the applied force. When the force acts to the right, the system would also be displaced to the right.
2) If $\beta = \bar{\omega}/\omega \gg 1$ (i.e., the force is “rapidly varying”), ϕ is close to $180°$ and the displacement is essentially of opposite phase relative to the applied force. When the force acts to the right, the system would be displaced to the left.
3) If $\beta = \bar{\omega}/\omega = 1$ (i.e., the forcing frequency is equal to the natural frequency), ϕ = $90°$ for all values of ξ, and the displacement attains its peaks when the force passes through zeros.

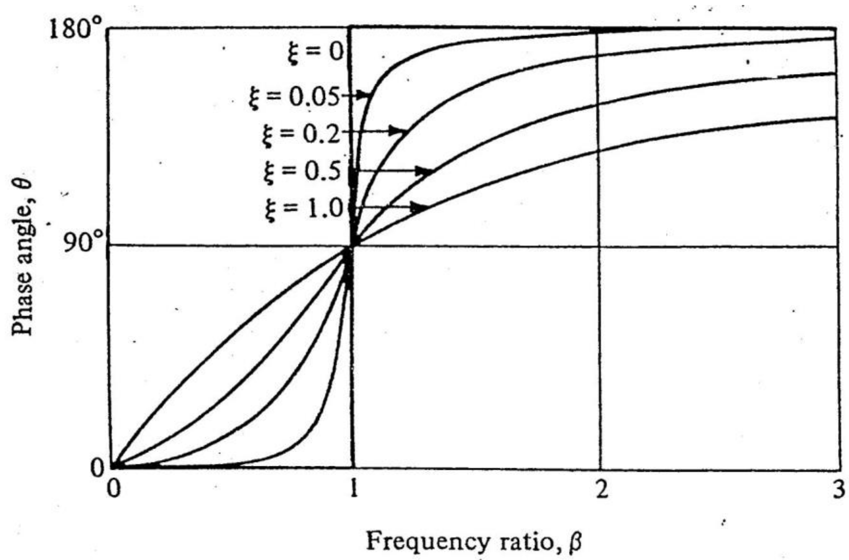


Figure 2-17: The relationship between frequency ratio and phase angle

Additional explanation on "pseudo-static", "inertial" and "resonant" ranges:

Let us consider the equation of motion.

$$f_I + f_D + f_s = p_o \sin(\omega t)$$

The left-hand side of the equation contains three structural dynamic forces. The right-hand side is an external force.

The proportion of these three forces (at steady-state condition) is derived as follows.

$$u(t) = \rho \sin(\overline{\omega} t - \theta)$$

$$\dot{u}(t) = \overline{\omega}\rho \cos(\overline{\omega} t - \theta)$$

$$\ddot{u}(t) = -\overline{\omega}^2\rho \sin(\overline{\omega} t - \theta)$$

$u(t)$ and $\ddot{u}(t)$ are in opposite phase.

$$f_s = ku$$

$$|f_s|_{\max} = k\rho$$

$$(f_s)_n = \frac{|f_s|_{\max}}{k\rho} = 1$$

$$f_D = c\dot{u}$$

$$|f_D|_{\max} = 2\xi\beta k\rho$$

$$(f_D)_n = \frac{|f_D|_{\max}}{k\rho} = 2\xi\beta$$

$$f_I \ = m\ddot{u}$$

$$|f_I|_{\max} = \ \beta^2 k\rho$$

$$(f_I)_n \ = \frac{|f_s|_{\max}}{k\rho} = \beta^2$$

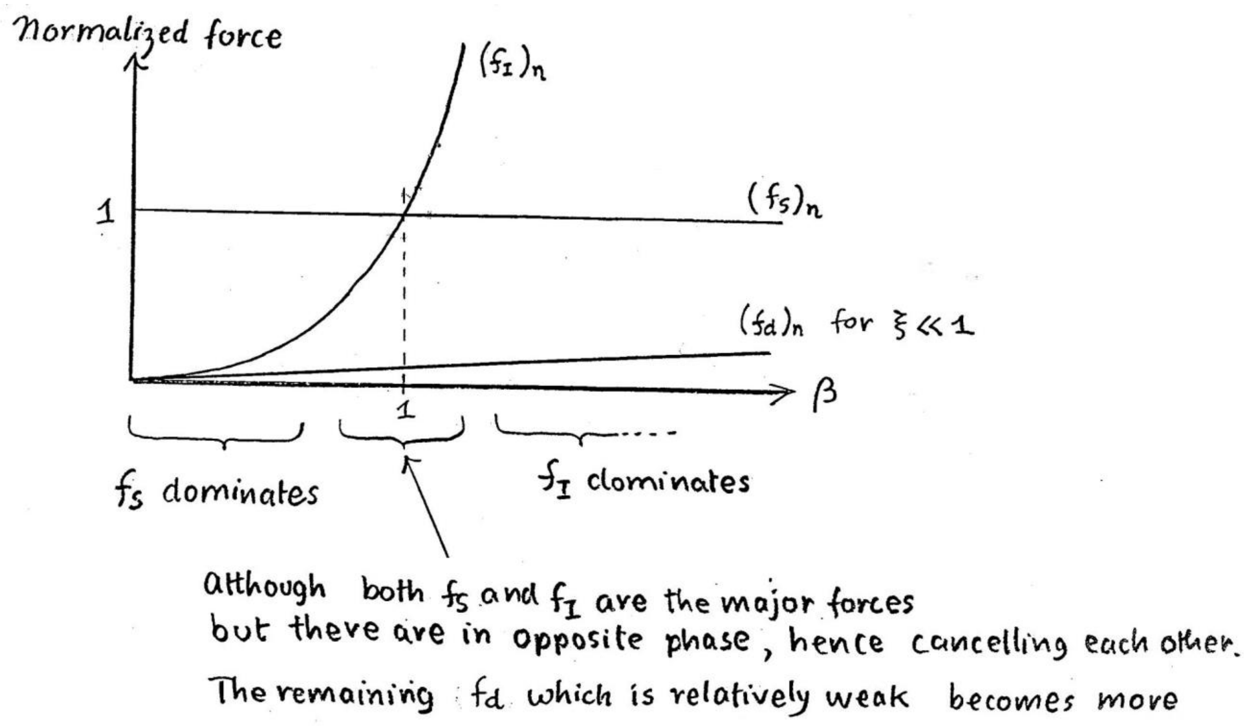


Figure 2-18: The “pseudo-static”, “inertial” and “resonant” ranges

Close to $\beta = 1$, although both f_s and f_I are the major forces but they are in opposite phase, hence cancelling each other. The remaining f_D which is relatively weak force becomes more important in this middle range.

At $\beta = 1$, the equation of motion becomes $f_D = p_o \sin(\omega t)$. In order to satisfy this equilibrium, large ρ is developed → resonance.

$$|f_D|_{\max} = \ 2\xi\beta k\rho$$

The term 2ξ is small, and ρ is Large.

2.5.4. Resonant Response of Damped Systems

To gain more understanding in the nature of resonant response, let us consider the general solution $u(t)$ at $\beta = 1$.

$$\rho_p = \frac{p_o}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} = \frac{p_o}{k}\frac{1}{2\xi}$$

$$R_D = \frac{1}{2\xi}$$

$$\rho_p = \frac{p_o}{k}R_D = u_o^{st}\ R_D = \frac{u_o^{st}}{2\xi}$$

Similarly, for $\beta = 1$,

$$\theta_p = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) = \frac{\pi}{2}$$

$$u_p(t) = \frac{p_o}{k}\left(\frac{1}{2\xi}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$

Therefore, the general solution becomes,

$$u(t) = e^{-\xi\omega t}\ \rho_h \cos(\omega t - \theta_h) + \frac{p_o}{k}\left(\frac{1}{2\xi}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$

$$\sin(\omega t - \frac{\pi}{2}) = -\cos(\omega t)$$

Assume that the structure initial has no motion i.e. $u(0) = 0$ and $\dot{u}(0) = 0$. With these specified initial conditions, ρ_h and θ_h can be determined and, we finally obtain

$$u(t) = \frac{1}{2\xi}\ \frac{p_o}{k}\left(e^{-\xi\omega t}\left[\cos\omega_D t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D t\right] - \cos\omega t\right)$$

For lightly damped systems the $\omega_D = \omega$; thus

$$u(t) \cong \frac{1}{2\xi}\ \frac{p_o}{k}\ \left(e^{-\xi\omega t} - 1\right)\cos\omega t$$

The term $\frac{1}{2\xi}\ \frac{p_o}{k}\ \left(e^{-\xi\omega t} - 1\right)$ is the envelope function.

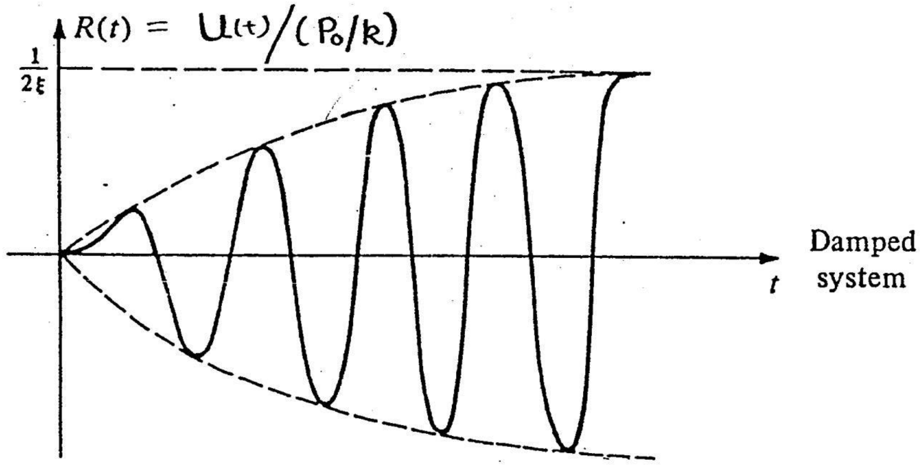


Figure 2-19: The response to resonant loading $\beta = 1$ for at-rest initial conditions. The response builds up gradually until the amplitude approaches the steady-state value.

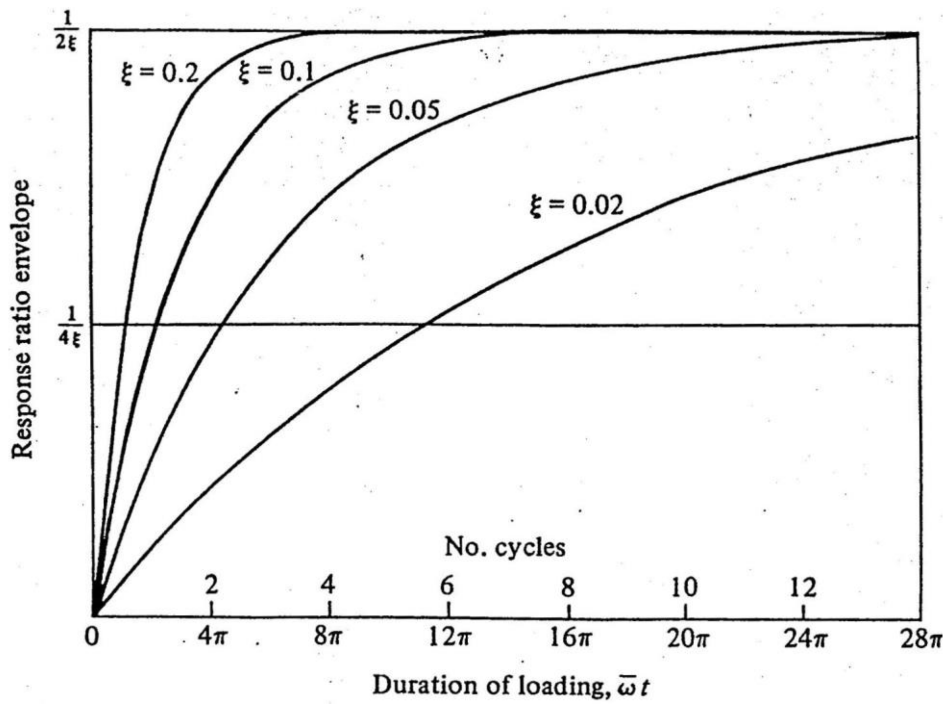


Figure 2-20: The rate of buildup of resonant response from rest.

Different structures take different number of cycles to converge to large response of resonance e.g. for R_D = 10, to reach about 90% to 95% of steady-state amplitude, we need approximately 10 cycles. If R_D = 20, the structure would require 20 cycles. So resonance just not happens immediately. If loading is comprised of just few cycles, the structure would not produce large dynamic amplitude.

The term $\frac{1}{2\xi}\frac{p_o}{k}\left(e^{-\xi\omega t}-1\right)$ is the envelope function. The value $\left(e^{-\xi\omega t}-1\right)$ starts from 0 and approach -1 for large values of t.

For highly damped systems, it takes only a few cycles to reach the peak. For lowly damped systems, it may take large number of cycles to reach the peak.

Therefore, in order that large resonant response to be careful develop, three conditions have to be met:

a. Frequency Tuning, $\beta = 1$
b. Low damming ratio, $\xi \ll 1$
c. Sufficiently long duration of excitation

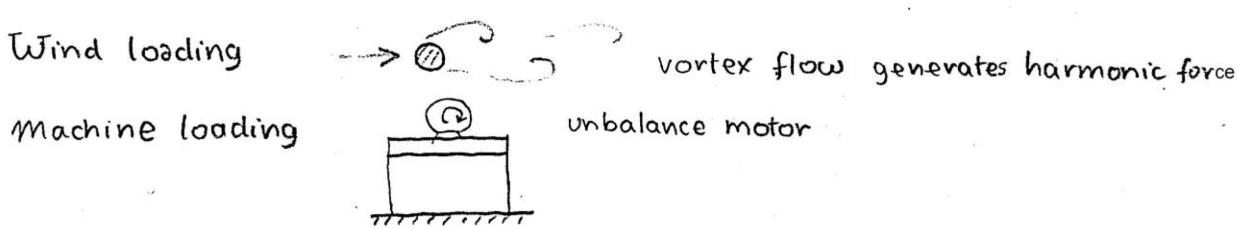


Figure 2-21: The examples of harmonic loading which can cause resonant response

2.5.5. Dynamic Response Factors

The steady-state response of an SDF system subjected to harmonic loading $\mathrm{p_o}\sin(\overline{\omega}\mathrm{t})$ is shown again.

$$u(t) = \rho_p \sin(\overline{\omega}t - \theta_p)$$

Where

$$\rho_p = \frac{p_o}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} = u_o^{st}\, R_D$$

$$\theta_p = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

$$R_D = \frac{\rho_p}{u_o^{st}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

$$u(t) = u_o^{st} R_D \sin(\overline{\omega}t - \theta_p)$$

R_D is the displacement response factor. By differentiating above equation, we can also get velocity and acceleration response factors as follows.

$$R_V = \frac{\overline{\omega}}{\omega} R_D = \beta R_D$$

$$R_A = \left(\frac{\overline{\omega}}{\omega}\right)^2 R_D = \beta^2 R_D$$

$$R_A = \frac{\left(\frac{f'}{f}\right)^2}{\sqrt{\left(1-\left(\frac{\bar{f}}{f}\right)^2\right)^2 + \left(2\xi\frac{\bar{f}}{f}\right)^2}}$$

R_A is also a function of frequency ratio ($\bar{f}/f$) and damping ratio (ξ).

Therefore,

$$\frac{R_A}{\beta} = R_V = \beta R_D$$

A resonant frequency is defined as the forcing frequency at which the largest response amplitude occurs. The peaks in the frequency-response curves for displacement, velocity, and acceleration occur at slightly different frequencies. These resonant frequencies can be determined by setting to zero the first derivative of R_D, R_V, and R_A with respect to β; for $\xi < 1/\sqrt{2}$ they are:

Displacement resonant frequency:

Velocity resonant frequency: ω

Acceleration resonant frequency:

For an undamped system the three resonant frequencies are identical and equal to the natural frequency ω n of the system. Intuition might suggest that the resonant frequencies for a damped system should be at its natural frequency $\omega_D = \omega\sqrt{1-\xi^2}$, but this does not happen; the difference is small, however. For the degree of damping usually embodied in structures, typically well below 20%, the differences among the three resonant frequencies and the natural frequency are small.

The three dynamic response factors at their respective resonant frequencies are

$$R_D = \frac{1}{2\xi\sqrt{1-\xi^2}} = R_A$$

$$R_V = \frac{1}{2\xi}$$

2.5.6. Response to Harmonic Ground Motion

Harmonic ground motion is represented by

$$u_g(t) = u_{go}\sin(2\pi\bar{f}t)$$

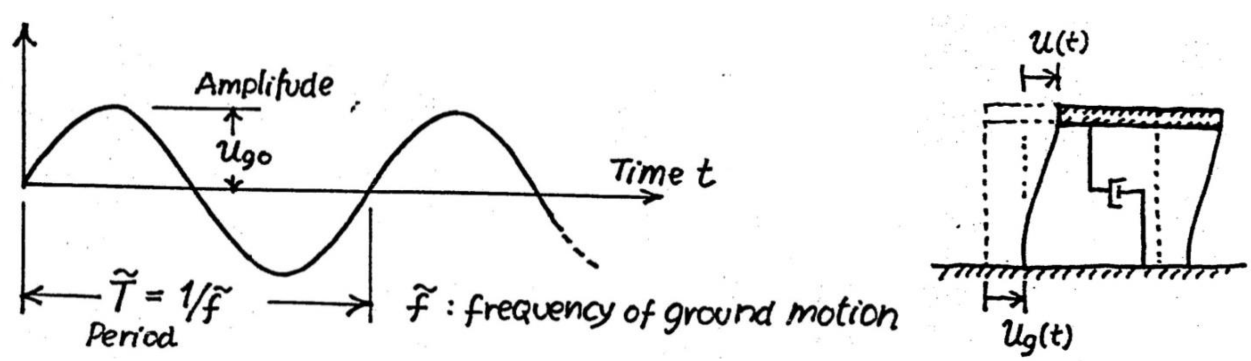


Figure 2-23: The harmonic ground motion

Effective Force:

$$P_{eff}(t) = -m\frac{d^2u}{dt^2} = m\left(2\pi\bar{f}\right)^2 u_{go}\sin(2\pi\bar{f}t)$$

Equation of motion:

$$\frac{md^2u}{dt^2} + c\frac{du}{dt} + ku = P_{eff}(t) = m\left(2\pi\bar{f}\right)^2 u_{go}\sin(2\pi\bar{f}t)$$

Response:

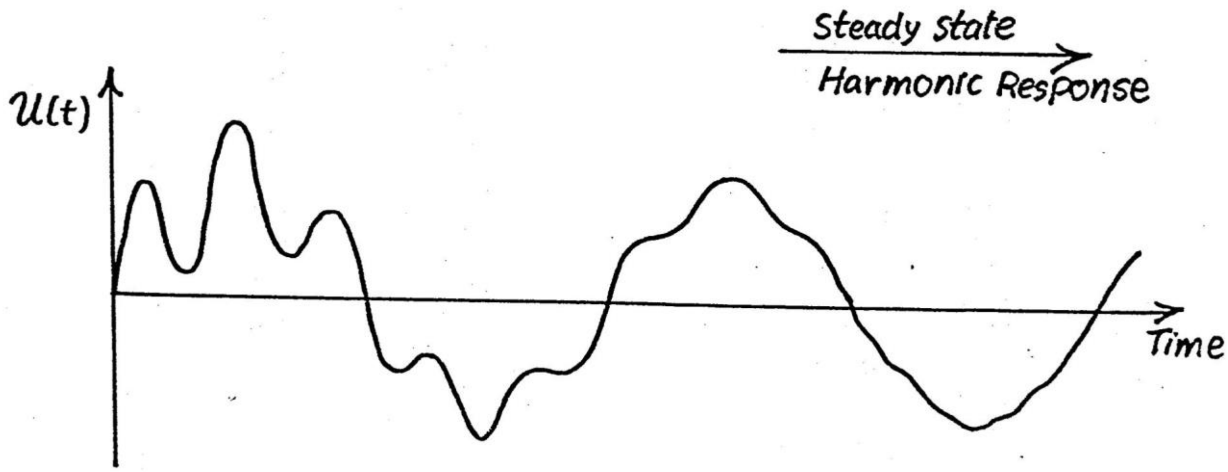


Figure 2-24: The response to harmonic ground motion

At Steady stage:

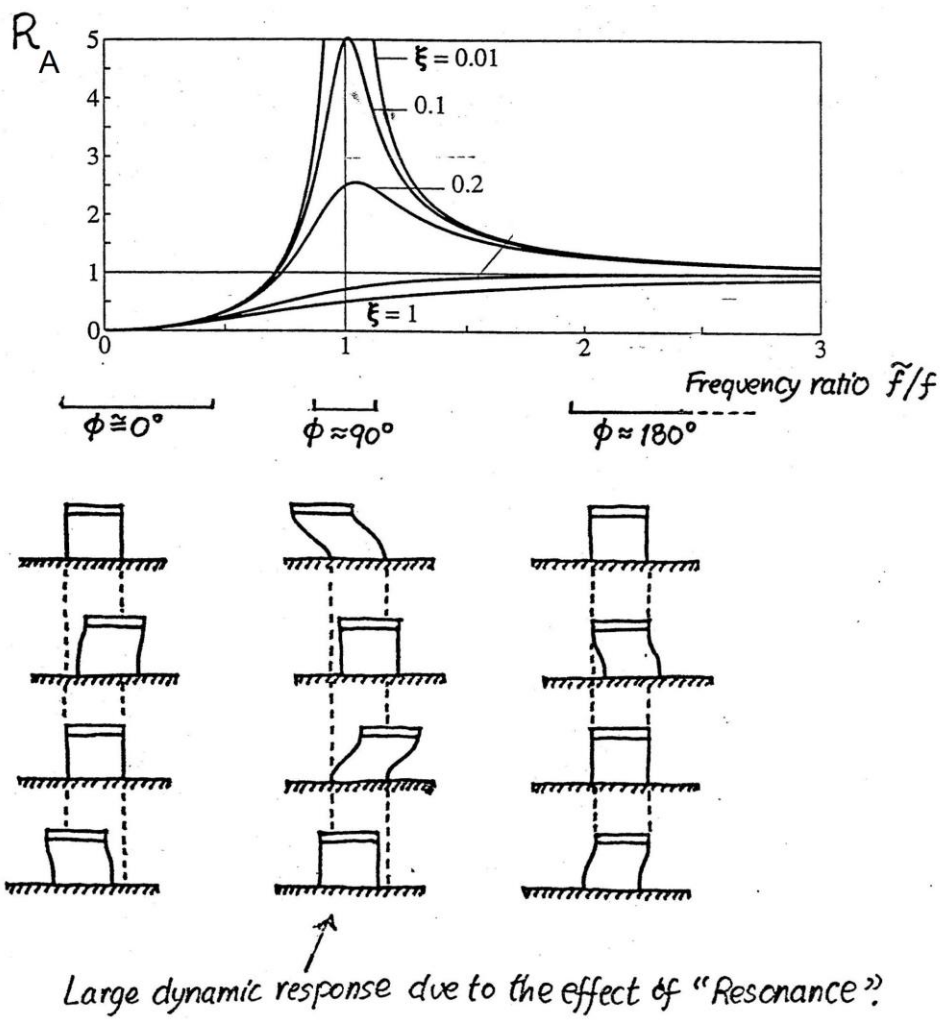
$$u(t) = R_D\, u_{go}\sin(2\pi\bar{f}t - \phi)$$

ϕ = phase lag

R_D = Dynamic amplification factor

For $R_D = 1$, the structure will have same amplitude of shaking as the ground shaking.

The same ground shaking is not equally harmful to all structures because they will have different natural frequencies and therefore, respond differently.



(a) (b) (c)

Figure 2-26: The response relation between frequency ratio and dynamic response factor

In (a), the structure is in-phase with ground shaking, but have low amplitude. In (b), the response of structure lags by a quarter-cycle to the ground shaking. In (c), the mass of structure remains at the same place due to high inertial force, while the ground shakes. The structural response and ground shaking are completely out of phase.

2.5.7. Solved Examples

Harmonic Response of an SDF System

Example 5: For the same SDF system shown in Example 1 (damped case), determine the forced vibration response under a harmonic force defined as $p_o \sin \bar{\omega} t$ where $p_o = 1000\, N$ and $\bar{\omega} = 2\pi\bar{f}$. Consider the following three cases.

a) $\bar{f} = 0.5\, f$
b) $\bar{f} = f$

c) $\bar{f} = 2\,f$

Use at-rest initial conditions (i.e. $u(0) = 0,\ \dot{u}(0) = 0$). Plot the response in each case and find maximum displacement, base shear and base moment.

2.5.8. Estimation of Damping using Harmonic Tests

The theory of forced harmonic vibration, presented in the preceding sections provides a basis to determine the natural frequency and damping of a structure from its measured response to a vibration generator. The measured damping provides data for an important structural property that cannot be computed from the design of the structure. The measured value of the natural frequency is the "actual" property of a structure against which values computed from the stiffness and mass properties of structural idealizations can be compared. Such research investigations have led to better procedures for developing structural idealizations that are representative of actual structures (Taken from Chopra (2012) Dynamics of Structures, 4th Edition).

Resonance makes dynamic response much different from static response. "Resonant magnification" is governed by "damping". But it is usually not easy to determine the damping c for a given structure. In fact, it is a major source of error in dynamic analysis. So, the value of c is usually assumed based on past experiences.

Damping coefficient c can be evaluated directly from experiments. One common technique is "free vibration decay". It was discussed in free vibration response.

Resonant Amplification Method

Another technique is to estimate c from frequency-response curve. This method of determining the viscous-damping ratio is based on measuring the steady-state amplitudes of relative-displacement response produced by separate harmonic loadings of amplitude po at discrete values of excitation frequency ω over a wide range including the natural frequency. Plotting these measured amplitudes against frequency provides a frequency-response curve of the type shown in Fig. 3-15.

Since the peak of the frequency-response curve for a typical low damped structure is quite narrow, it is usually necessary to shorten the intervals of the discrete frequencies in the neighborhood of the peak in order to get good resolution of its shape.

The damping ratio can then be determined from the experimental data using

$$\xi = \frac{1}{2R_D}$$

Half-Power (Band-Width) Method

An important property of the frequency response curve for R_d shown below is the half-power bandwidth. If ω_a and ω_b are the forcing frequencies on either side of the resonant frequency at which the amplitude u_o is $1/\sqrt{2}$ times the resonant amplitude, then for small ξ, it can be shown that,

$$\frac{\omega_b - \omega_a}{\omega} = 2\xi$$

$$\xi = \frac{\omega_b - \omega_a}{2\omega}$$

Half power band width = 2ξ

This important result enables evaluation of damping from forced vibration tests without knowing the applied force.

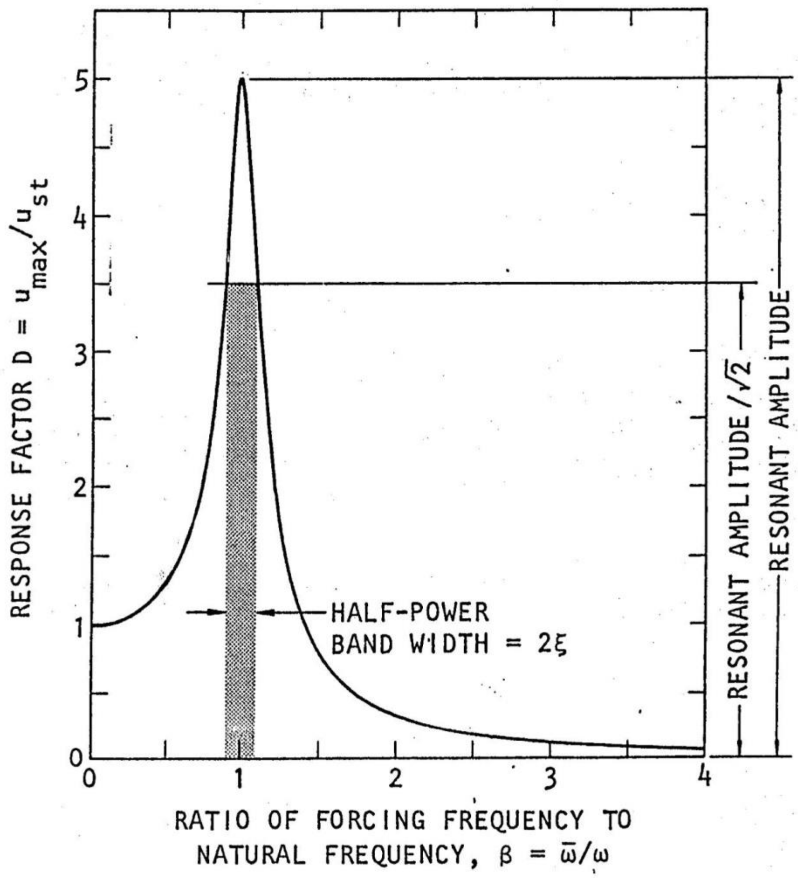


Figure 2-22: The Evaluation of damping from force vibration tests

The damping can also be determined using the resonance testing. The basic idea is

$$\xi = \frac{1}{2}\frac{u_o^{st}}{u_{o(\bar{\omega}=\omega)}}$$

2.5.9. Summary - Response to Harmonic Force

Equation of Motion:

$$\frac{md^2u}{dt^2} + c\frac{du}{dt} + ku = p_o sin(2\pi\bar{f}t)$$

Response (at the steady state):

$$u(t) = \frac{p_o}{k}\, R_D \sin(2\pi \bar{f} t - \theta_p)$$

Where $\frac{p_o}{k}$ = static response to a static force p_o.

θ_p = Phase lag

$D =$ Dynamic amplification factor

$$D = \frac{1}{\sqrt{\left(1 - \left(\frac{\bar{f}}{f}\right)^2\right)^2 + \left(2\xi \frac{\bar{f}}{f}\right)^2}}$$

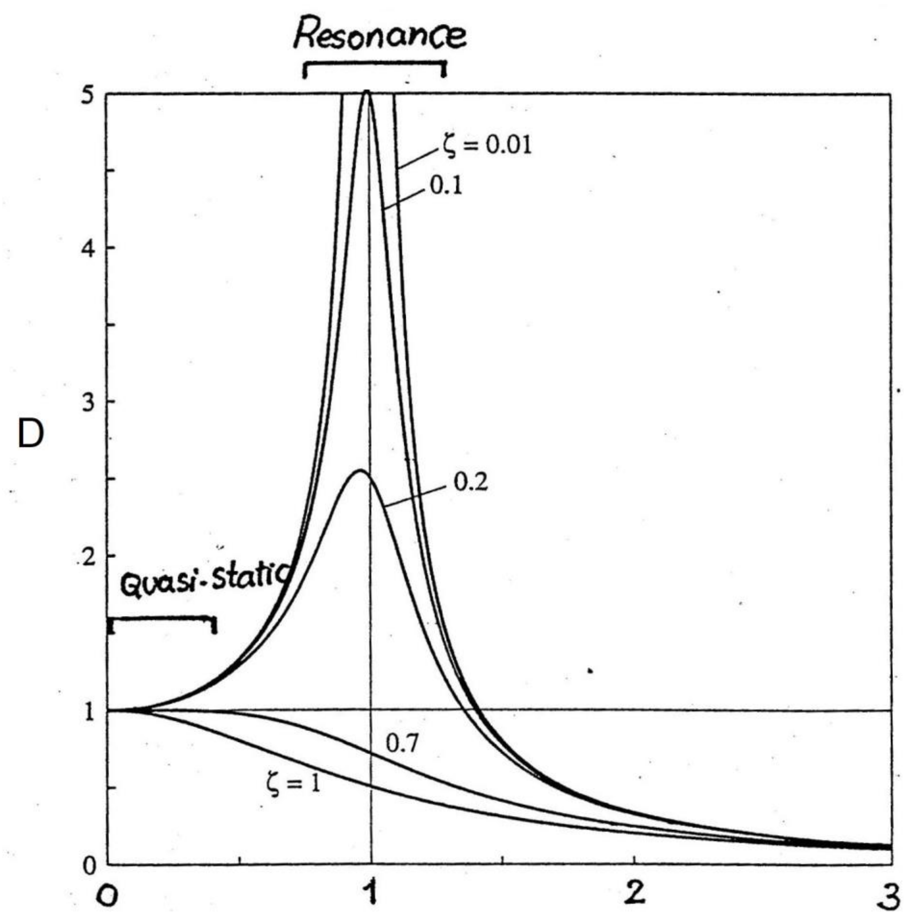


Figure 2-25: The response relation between frequency ratio and dynamic amplification factor

2.5.10. Steady-state Response to Cosine Force ($p_o cos\bar{\omega}t$)

The response to cosine loading can also be found in similar manner. In this case, the coefficients of particular solution (G_1 and G_2) will be as follows.

$$G_1 = \frac{p_o}{k} \frac{2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

$$G_2 = \frac{p_o}{k} \frac{1-\beta^2}{(1-\beta^2)^2 + (2\xi\beta)^2}$$

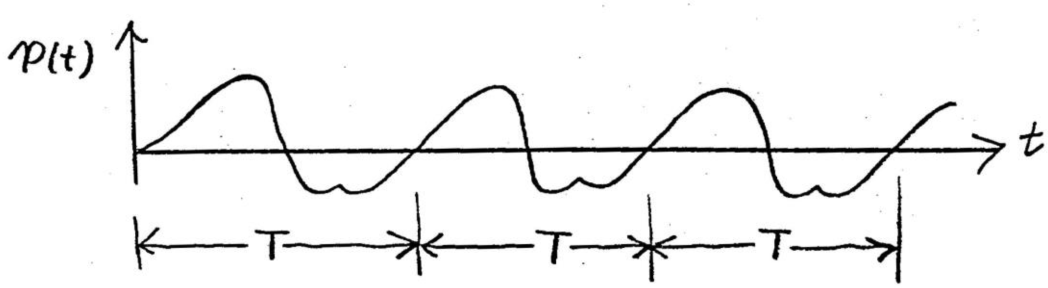
The steady-state response would be

$$u(t) = \frac{p_o}{k} R_D \cos(\bar{\omega}t - \theta_p) = u_o^{st} R_D \cos(\bar{\omega}t - \theta_p)$$

Where R_D and θ_p are the same as derived for sinusoidal force. This similarity in the steady-state responses to the two harmonic forces is not surprising since the two excitations are the same except for a time shift.

2.6. Response to Periodic Loading

A periodic function is one in which the portion defined over T repeats itself indefinitely as shown in the figure below. Many forces are periodic or nearly periodic. Under certain conditions, propeller forces on a ship, wave loading on an offshore platform, and wind forces induced by vortex shedding on tall, slender structures are nearly periodic.



A periodic function $p(t)$ with a period T

2.6.1. Fourier Series Representation of a Periodic Function

Any arbitrary periodic functions can be represented in terms of a summation of simple sine and cosine functions.

$$p(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\bar{\omega}t) + \sum_{n=1}^{\infty} b_n \sin(n\bar{\omega}t)$$

Where $\bar{\omega} = 2\pi/T$.

The right hand side of the above expression is called “Fourier series”, i.e. a periodic function can be separated into its harmonic components using the Fourier series.

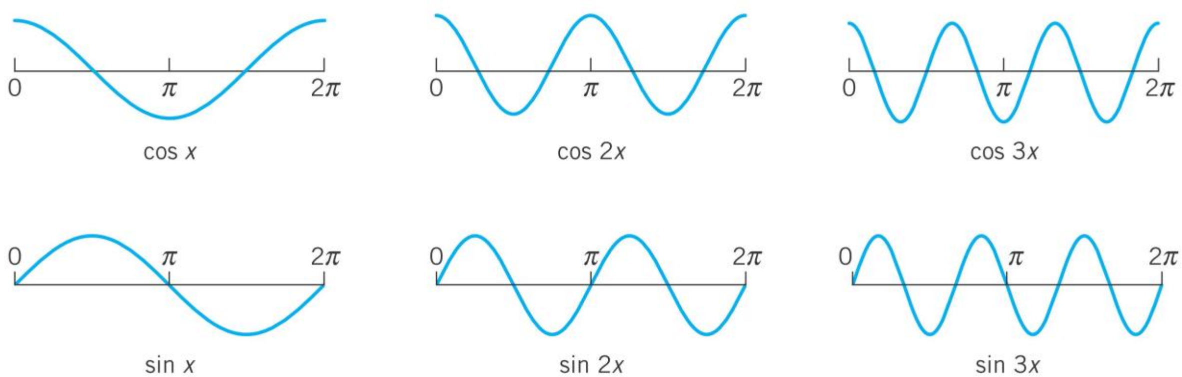
This concept called Fourier decomposition was first proposed by Jean-Baptiste Joseph Fourier, a French physicist and mathematician (1768 - 1830), lived and taught in Paris, accompanied Napoléon in the Egyptian War, and was later made prefect of Grenoble. The beginnings on Fourier series can be found in works by Euler and by Daniel Bernoulli, but it was Fourier who employed them in a systematic and general manner in his main work, Théorie analytique de la chaleur (Analytic Theory of Heat, Paris, 1822), in which he developed the theory of heat conduction (heat equation; see Sec. 12.5), making these series a most important tool in applied mathematics (Erwin Kreszig, Advanced Engineering Mathematics, 10th Edition).

The central starting point of Fourier analysis is Fourier series. They are infinite series designed to represent general periodic functions in terms of simple ones, namely, cosines and sines. This trigonometric system is orthogonal, allowing the computation of the coefficients of the Fourier series by use of the well-known Euler formulas. Fourier series are, in a certain sense, more universal than the familiar Taylor series in calculus because many discontinuous periodic functions that come up in applications can be developed in Fourier series but do not have Taylor series expansions.

The underlying idea of the Fourier series can be extended in two important ways. We can replace the trigonometric system by other families of orthogonal functions, e.g., Bessel functions and obtain the Sturm–Liouville expansions. The second expansion is applying Fourier series to nonperiodic phenomena and obtaining Fourier integrals and Fourier transforms.

In a digital age, the discrete Fourier transform plays an important role. Signals, such as voice or music, are sampled and analyzed for frequencies. An important algorithm, in this context, is the fast Fourier transform. Note that the two extensions of Fourier series are independent of each other.

Fourier analysis allows us to model periodic phenomena which appear frequently in engineering and elsewhere—think of rotating parts of machines, alternating electric currents or the motion of planets. Related period functions may be complicated. Now, the ingenious idea of Fourier analysis is to represent complicated functions in terms of simple periodic functions, namely cosines and sines. The representations will be infinite series called Fourier series. This idea can be generalized to more general Fourier series and to Fourier integral (Erwin Kreszig, Advanced Engineering Mathematics, 10th Edition).



Cosine and sine functions having the period 2π

If $p(t)$ is given, the coefficients a_n and b_n can be determined by simple integrations as follows. The expressions are called Euler formulas.

$$\int_{t=0}^{t=T} p(t)dt = \int_{t=0}^{t=T} \left[a_o + \sum_{n=1}^{\infty} a_n \cos(n\bar{\omega}t) + \sum_{n=1}^{\infty} b_n \sin(n\bar{\omega}t) \right] dt = a_o T$$

$$a_o = \frac{1}{T} \int_{t=0}^{t=T} p(t)dt$$

Similarly,

$$\int_{t=0}^{t=T} p(t)\cos(m\bar{\omega}t)\,dt = \int_{t=0}^{t=T}\left[a_o + \sum_{m=1}^{\infty} a_m \cos(m\bar{\omega}t) + \sum_{m=1}^{\infty} b_m \sin(m\bar{\omega}t)\right]\cos(m\bar{\omega}t)\,dt = \frac{a_m T}{2}$$

$$a_m = \frac{2}{T}\int_{t=0}^{t=T} p(t)\cos(m\bar{\omega}t)\,dt$$

Similarly, it can be shown that,

$$b_m = \frac{2}{T}\int_{t=0}^{t=T} p(t)\sin(m\bar{\omega}t)\,dt$$

Orthogonal Vectors and Orthogonal Functions:

$\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal vectors if, $\boldsymbol{a}.\boldsymbol{b} = \mathbb{0}$, **or** $[\boldsymbol{a}]^{\boldsymbol{t}}[\boldsymbol{b}] = 0$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are any functions of time t, they will be orthogonal functions if,

$$\int_{t=0}^{t=T} a(t)\, b(t)\, dt$$

Fourier series is a series of orthogonal functions i.e.

$$\int_{t=-T/2}^{t=T/2} \cos(nx)\cos(mx)\ \ dx = 0,\ \ n \neq m$$

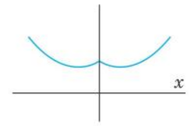
$$\int_{t=-T/2}^{t=T/2} \sin(nx)\sin(mx)\ \ dx = 0,\ \ n \neq m$$

$$\int_{t=-T/2}^{t=T/2} \sin(nx)\cos(mx)\ \ dx = 0,\ \ n \neq m$$

Simplifications: Even and Odd Functions

If $f(x)$ is an **even function**, that is, $f(-x) = f(x)$, its Fourier series reduces to a **Fourier cosine series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$



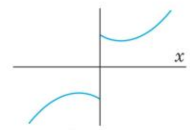
Even function

with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1, 2, \cdots$$

If $f(x)$ is an **odd function**, that is, $f(-x) = -f(x)$ its Fourier series reduces to a **Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$



Odd function

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Summary

Even Function of Period 2π. If f is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx, \qquad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx, \quad n = 1, 2, \cdots$$

Odd Function of Period 2π. If f is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

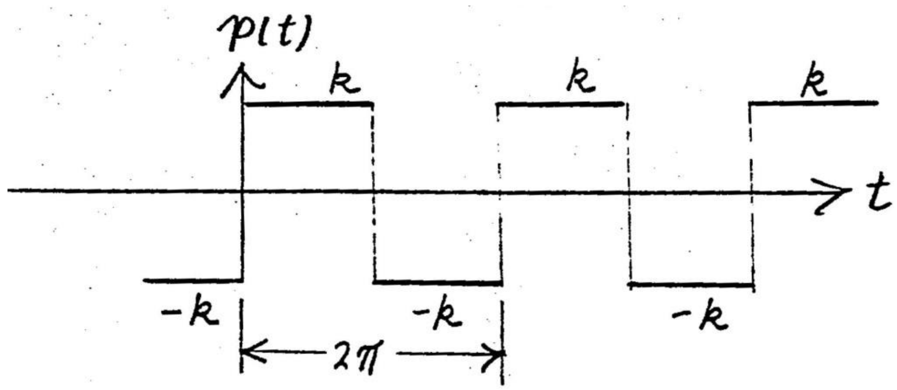
with coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx, \qquad n = 1, 2, \cdots.$$

Example:

Consider a periodic square function as shown below.

$$p(t) = \begin{Bmatrix} k & for\ 0 < t < \pi \\ -k & for\ \pi < t < 2\pi \end{Bmatrix}$$



A periodic square function

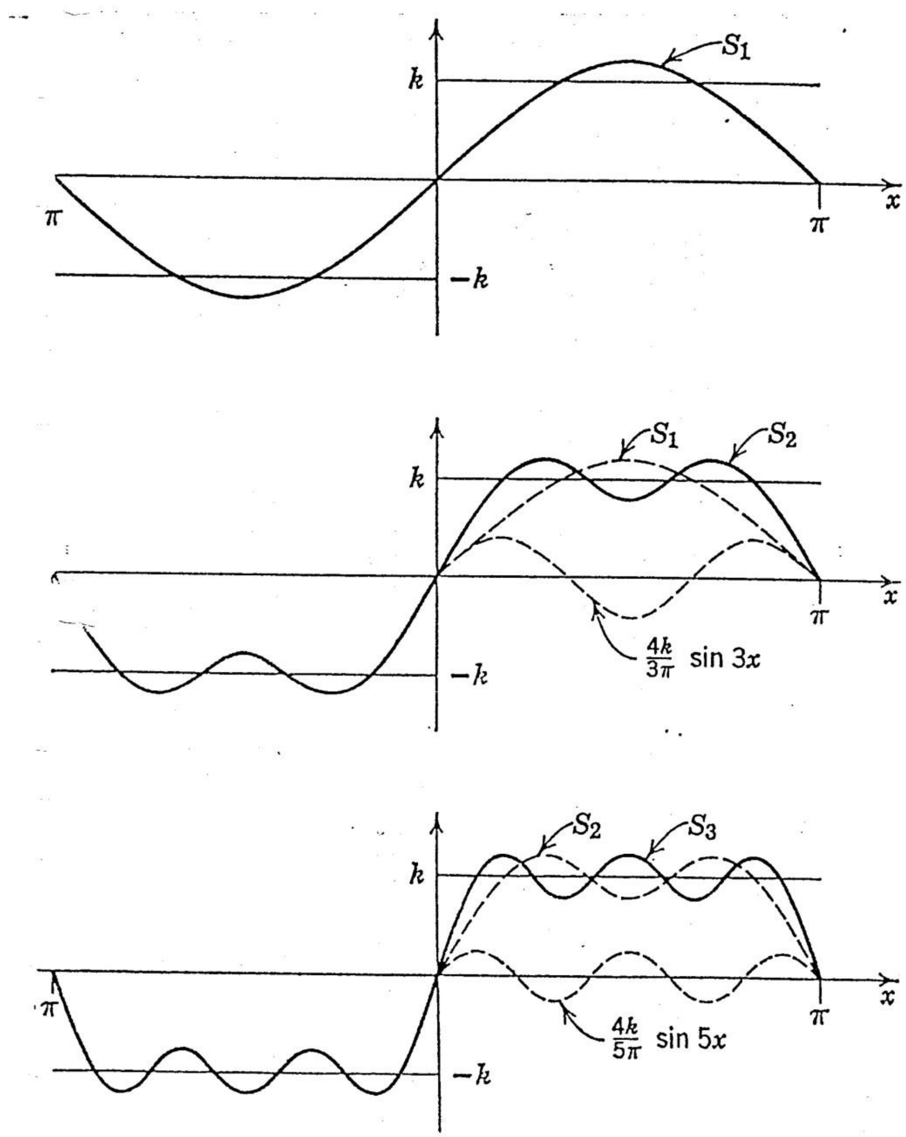
Conducting the integrations using the above equations, we obtain,

$$a_o = 0, \qquad a_n = 0, \qquad n = 1, 2, .3, \dots \infty$$

$$b_n = \frac{2k}{n\pi}(1 - \cos n\pi)$$

This is,

$$b_1 = \frac{2k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \dots$$



The first three partial sums of the corresponding Fourier series of the given square periodic function

The series coverage quickly to the square function.

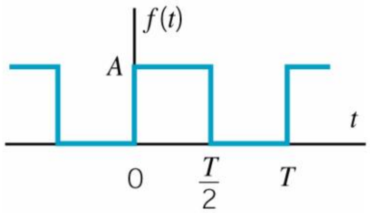
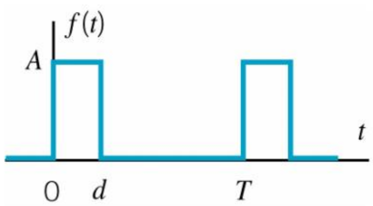
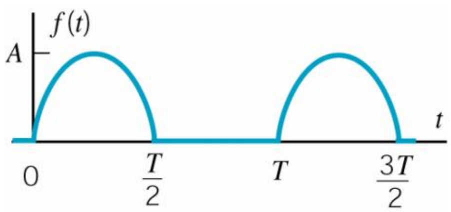
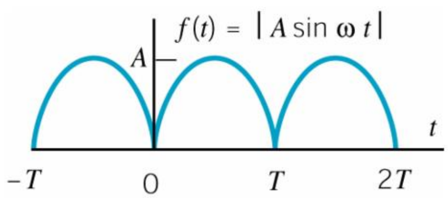
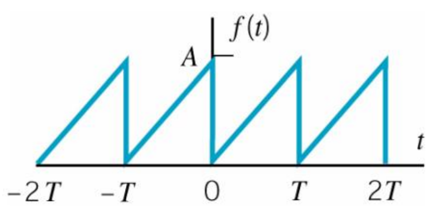
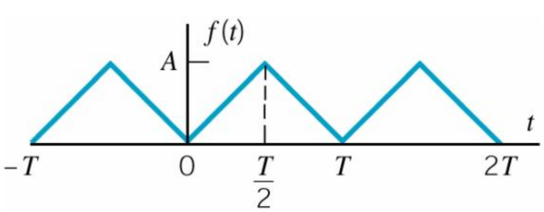
Theoretically, an infinite number of terms are required for the Fourier series to converge to $p(t)$. In practice, however, a few terms are sufficient for good convergence. At a discontinuity, the Fourier series converges to a value that is the average of the values immediately to the left and to the right of the discontinuity.

Therefore in many practical applications, it is not necessary to evaluate ∞ series. Only a finite series is good enough:

$$p(t) \approx \sum_{n=1}^{N} b_n \sin(n\bar{\omega}t)$$

When N is large but not ∞. Since $b_i = 2k/i\pi$ for $i = 1, 3, 5, ...$, we can also write the function $p(t)$ as

$$p(t) \approx \frac{4k}{\pi} \sum_{n=1,3,5}^{N} \frac{1}{n} \sin(n\overline{\omega}t)$$

Function	Trigonometric Fourier Series
	Square wave: $\omega_0 = \frac{2\pi}{T}$ $f(t) = \frac{A}{2} + \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left((2n-1)\omega_0 t\right)}{2n-1}$
	Pulse wave: $\omega_0 = \frac{2\pi}{T}$ $f(t) = \frac{Ad}{2} + \frac{2Ad}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi d}{T}\right)}{\frac{n\pi d}{T}} \cos\left(n\omega_0 t\right)$
	Half wave rectified sine wave: $\omega_0 = \frac{2\pi}{T}$ $f(t) = \frac{A}{\pi} + \frac{A}{2} \sin \omega_0 t - \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{\cos\left(2n\omega_0 t\right)}{4n^2 - 1}$
	Full wave rectified sine wave: $\omega_0 = \frac{2\pi}{T}$ $f(t) = \frac{2A}{\pi} - \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\cos\left(n\omega_0 t\right)}{4n^2 - 1}$
	Sawtooth wave: $\omega_0 = \frac{2\pi}{T}$ $f(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(n\omega_0 t\right)}{n}$
	Triangle wave: $\omega_0 = \frac{2\pi}{T}$ $f(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos\left((2n-1)\omega_0 t\right)}{(2n-1)^2}$

A periodic excitation implies that the excitation has been in existence for a long time, by which time the transient response associated with the initial displacement and velocity has decayed. Thus, we are interested in finding the steady-state response.

Response of a periodic loading = Response to the Fourier series of the loading

By Superposition, we can say that the response of a periodic loading = the sum of the responses to each sine and cosine loading in the series.

Superposition:

Let u_1 be response to $p_1(t)$ loading i.e.

$$m\ddot{u}_1 + c\dot{u}_1 + ku_1 = p_1(t)$$

And u_2 be the response to $p_2(t)$ i.e.

$$m\ddot{u}_2 + c\dot{u}_2 + ku_2 = p_2(t)$$

Then $u_1 + u_2$ is the response to $p_1(t) + p_2(t)$.

$$m(\ddot{u}_1 + \ddot{u}_2) + c(\dot{u}_1 + \dot{u}_2) + k(u_1 + u_2) = p_1(t) + p_2(t)$$

2.6.2. Steady-state Response to Periodic Loading

Consider an SDF structure is subjected to a periodic force.

$$p(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\bar{\omega}t) + \sum_{n=1}^{\infty} b_n \sin(n\bar{\omega}t)$$

$$u_{oa} = \frac{a_o}{k}$$

Define $\beta_n = n\bar{\omega}/\omega$ and use the result obtained from the previous section.

u_{bn} = steady-state response to $b_n \sin(n\bar{\omega}t)$.

$$u_{bn}(t) = \frac{b_n}{k}\frac{1}{\left(1-{\beta_n}^2\right)^2 + (2\xi\beta_n)^2}\left\{\left(1-{\beta_n}^2\right)\sin(n\bar{\omega}t) - 2\xi\beta_n\cos(n\bar{\omega}t)\right\}$$

$$u_{an}(t) = \frac{a_n}{k}\frac{1}{\left(1-{\beta_n}^2\right)^2 + (2\xi\beta_n)^2}\left\{2\xi\beta_n\sin(n\bar{\omega}t) + \left(1-{\beta_n}^2\right)\cos(n\bar{\omega}t)\right\}$$

The combined response would be,

$$u(t) = \frac{1}{k}\left[a^0 + \sum_{n=1}^{\infty}\frac{1}{\left(1-{\beta_n}^2\right)^2 + (2\xi\beta_n)^2}\left\{\left(a_n\,2\xi\beta_n + b_n\left(1-{\beta_n}^2\right)\right)\sin(n\bar{\omega}t) + \left(a_n\left(1-{\beta_n}^2\right) - b_n 2\xi\beta_n\right)\cos(n\bar{\omega}t)\right\}\right]$$

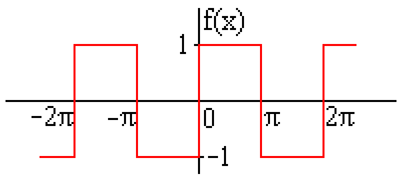
Example 1:

The steady-state response of an SDF structure subjected to the square periodic function described above would be

$$u(t) = \frac{p_o}{k}\frac{4}{\pi}\sum_{n=1,3,5}^{N}\frac{1}{n}\frac{1}{\left(1-\beta_n{}^2\right)^2+(2\xi\beta_n)^2}\left[\left(1-\beta_n{}^2\right)\sin(n\bar{\omega}t)-2\xi\beta_n\cos(n\bar{\omega}t)\right]$$

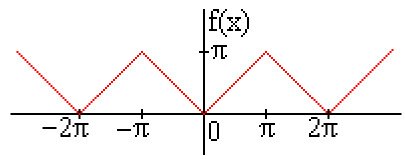
Example 2:

Show that



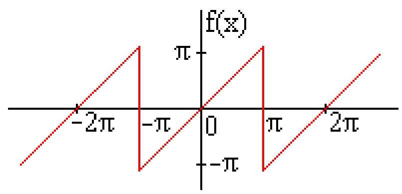
$$f(x) = |x| = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} =$$

$$= \frac{4}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)$$



$$f(x) = |x| = \begin{cases} x & 0 < x < \pi \\ -x & -\pi < x < 0 \end{cases} =$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right)$$



$$f(x) = x, \quad (-\pi < x < \pi) =$$

$$= 2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\right)$$

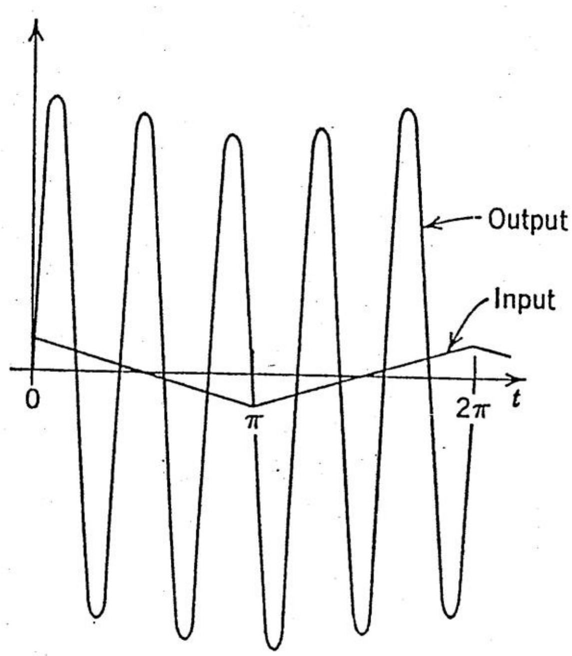
Example 3:

Response of an SDF structure with $\omega = 5$ rad/sec when subjected to a periodic loading of triangular waveform ($\bar{\omega} = 1$ rad/sec)

Inputs:

$\bar{\omega} = 1,\ \omega = 5,\ \beta_1 = \bar{\omega}/\omega = 0.2,\ \beta_3 = 3\bar{\omega}/\omega = 0.6\ ,\ \beta_5 = 5\bar{\omega}/\omega = 1, \ldots$

For β_5 term, the response will be dominated by resonance response at frequency $5\bar{\omega}$.

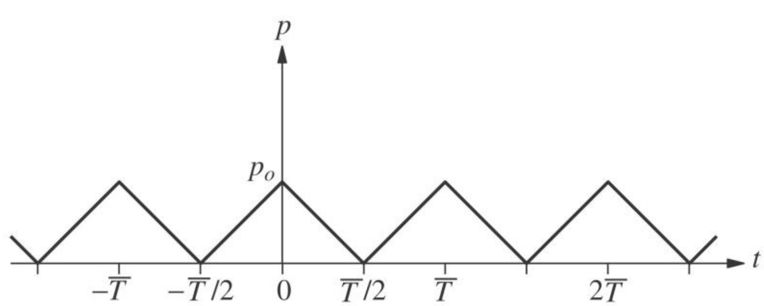


An example steady state response of an input triangular force

Example 4:

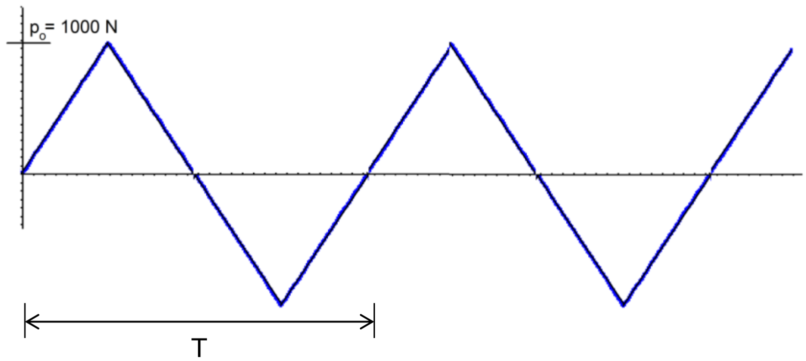
An SDF system with natural period T and damping ratio ξ is subjected to the periodic force shown in Figure below with an amplitude p_o and period $\bar{T}$.

a) Expand the forcing function in its Fourier series.
b) Determine the steady-state response of an undamped system.
c) For $\bar{T}/T$ = 2, determine and plot the response to individual terms in the Fourier series. How many terms are necessary to obtain reasonable convergence of the series solution?



Example 5:

An SDF system with mass $m = 2000\,Kg$, stiffness $k = 800000\,N/m$ and damping ratio $\xi = 0.027$ is subjected to the periodic force shown in Figure below.



Compute, plot and discuss the steady-state response and compare with previous example with harmonic loading. $\bar{\omega} = 2\pi\bar{f} = 2\pi/T$. Consider the same three cases.

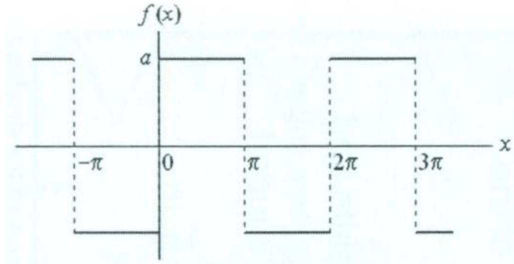
a) $\bar{f} = 0.5\,f$
b) $\bar{f} = f$
c) $\bar{f} = 2\,f$

Where f is the natural frequency of the structure. Use at-rest initial conditions (i.e. $u(0) = 0,\ \dot{u}(0) = 0$). Plot the response in each case and find maximum displacement, base shear and base moment.

Fourier series of some periodic functions

Function:

$$f(x) = \begin{cases} a & for\ 0 < x < \pi \\ -a & for - \pi < x < 0 \end{cases}$$

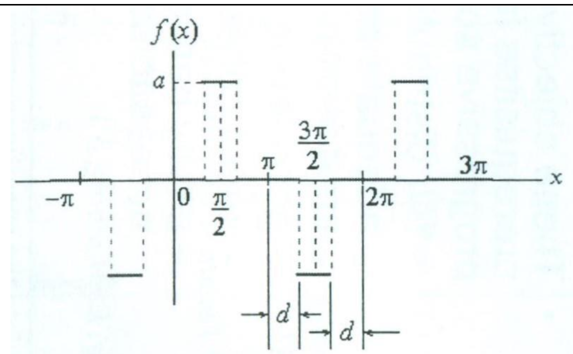


Fourier series:

$$f(x) = \frac{4a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Function:

$$f(x) = \begin{cases} a & for\ d < x < \pi - d \\ -a & for\ \pi + d < x < 2\pi - d \end{cases}$$

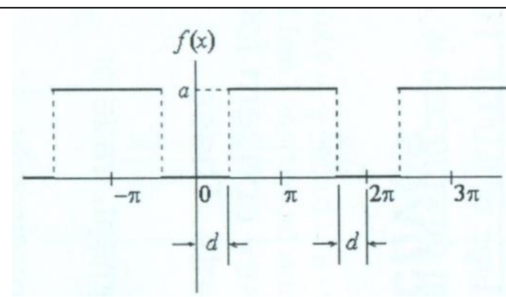


Fourier series:

$$f(x) = \frac{4a}{\pi}\left(\cos d \sin x + \frac{1}{3}\cos 3d \sin 3x + \frac{1}{5}\cos 5d \sin 5x + \cdots\right)$$

Function:

$$f(x) = \begin{cases} a & for\ d < x < 2\pi - d \\ 0 & for\ 0 < x < d\ and\ 2\pi - d < x < 2\pi \end{cases}$$

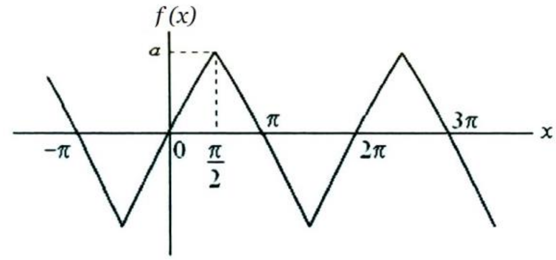


Fourier series:

$$f(x) = \frac{2a}{\pi}\left(\frac{\pi - d}{2} - \frac{\sin(\pi - d)}{1}\cos x + \frac{\sin 2(\pi - d)}{2}\cos 2x - \frac{\sin 3(\pi - d)}{3} + \cdots\right)$$

Function:

$$f(x)=\begin{cases}\dfrac{2ax}{\pi} & for-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}\\ \dfrac{2a(\pi-x)}{\pi} & for\ \dfrac{\pi}{2}\le x\le\dfrac{3\pi}{2}\end{cases}$$

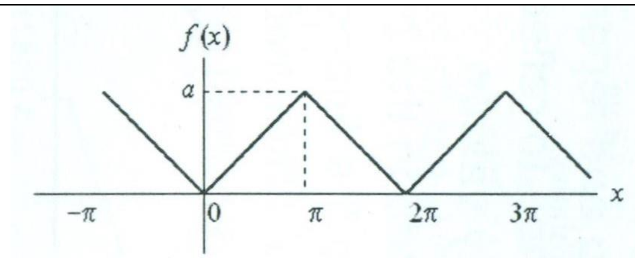


Fourier series:

$$f(x)=\frac{8a}{\pi^2}\left(\frac{\sin x}{1}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\cdots\right)$$

Function:

$$f(x)=\begin{cases}\dfrac{ax}{\pi} & for\ 0\le x\le\pi\\ \dfrac{a(2\pi-x)}{\pi} & for\ \pi\le x\le 2\pi\end{cases}$$

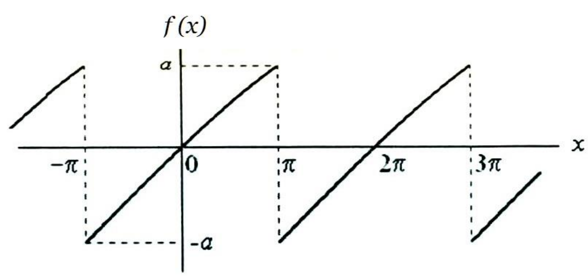


Fourier series:

$$f(x)=\frac{a}{2}-\frac{4a}{\pi^2}\left(\frac{\cos x}{1}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$$

Function:

$$f(x)=\left\{\frac{ax}{\pi}\quad for-\pi<x<\pi\right.$$

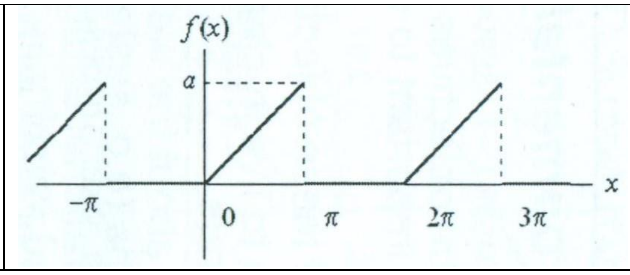


Fourier series:

$$f(x)=\frac{2a}{\pi}\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{\pi} & for\ 0 \le x \le \pi \\ 0 & for\ \pi \le x \le 2\pi \end{cases}$$

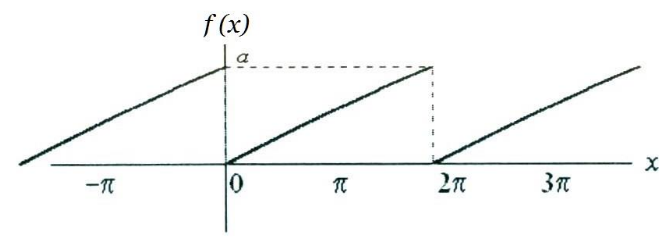


Fourier series:

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \frac{a}{\pi}\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{2\pi} & for\ 0 < x < 2\pi \end{cases}$$

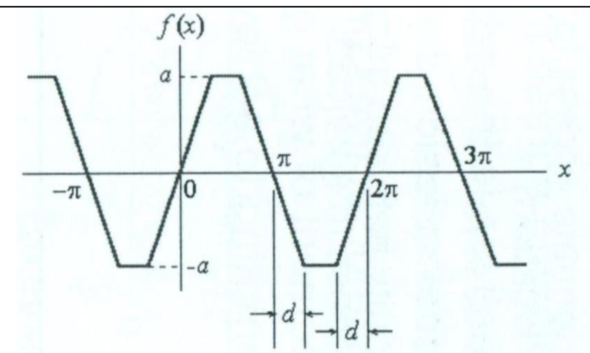


Fourier series:

$$f(x) = \frac{a}{2} - \frac{a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{d} & -b \le x \le b \\ a & b \le x \le \pi - b \\ \dfrac{a(\pi - x)}{d} & for\ \pi - b < x \le \pi + b \\ -a & for\ \pi + b < x \le 2\pi - b \end{cases}$$



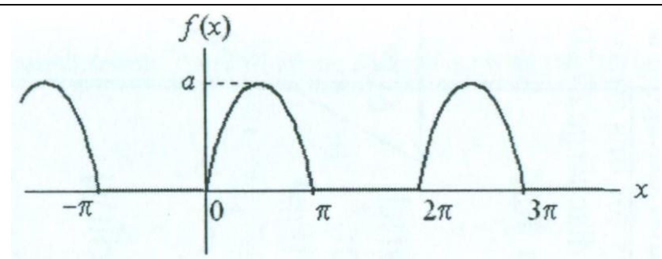
Fourier series:

$$f(x) = ?$$

Find yourself

Function:

$$f(x) = \begin{cases} a\sin x & for\ 0 \le x \le \pi \\ 0 & for\ \pi \le x \le 2\pi \end{cases}$$

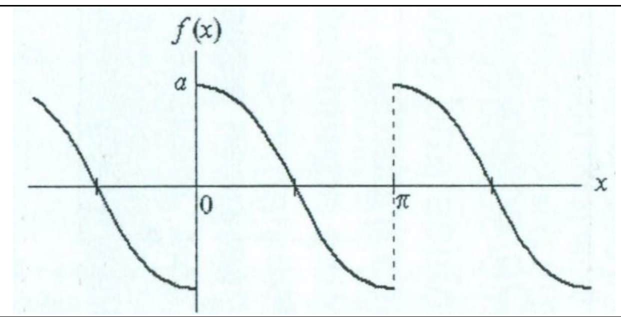


Fourier series:

$$f(x) = \frac{2a}{\pi}\left(\frac{1}{2} + \frac{\pi \sin x}{4} - \frac{\cos 2x}{1 \times 3} - \frac{\cos 4x}{3 \times 5} - \frac{\cos 6x}{5 \times 7} - \cdots\right)$$

Function:

$$f(x) = \begin{cases} a\cos x & for\ 0 < x < \pi \\ -a\cos x & for - \pi < x < 0 \end{cases}$$

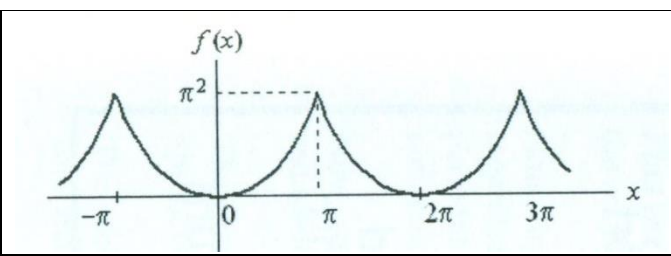


Fourier series:

$$f(x) = \frac{8a}{\pi}\left(\frac{\sin 2x}{1 \times 3} + \frac{2\sin 4x}{3 \times 5} + \frac{3\sin 6x}{5 \times 7} + \cdots\right)$$

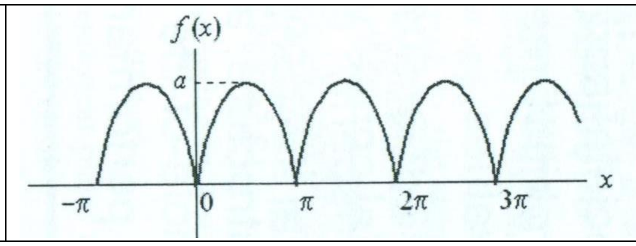
Function:

$$f(x) = \{x^2 \quad for\ -\pi \le x \le \pi$$



Fourier series:

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

Function: $$f(x) = \{a\lvert\sin x\rvert \quad for\ -\pi < x < \pi$$	
Fourier series: $$f(x) = \frac{2a}{\pi} - \frac{4a}{\pi}\left(\frac{\cos 2x}{1\times 3} + \frac{\cos 4x}{3\times 5} + \frac{\cos 6x}{5\times 7} + \cdots\right)$$	

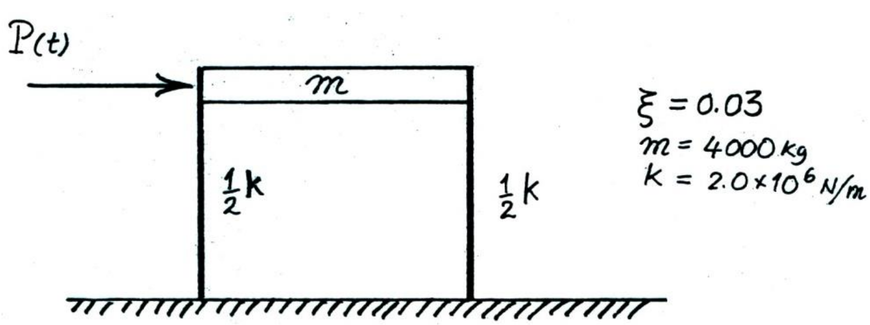
Mean Value Convergence Theorem:

If a periodic function $f(x)$ with period $2L$ is piecewise continuous over the interval $[-L, L]$, the Fourier Series of $f(x)$ converges to the ***mean value*** $\frac{1}{2}[f(x+)+f(x-)]$ at point x where both the left-hand and right-hand first derivatives of $f(x)$ exist.

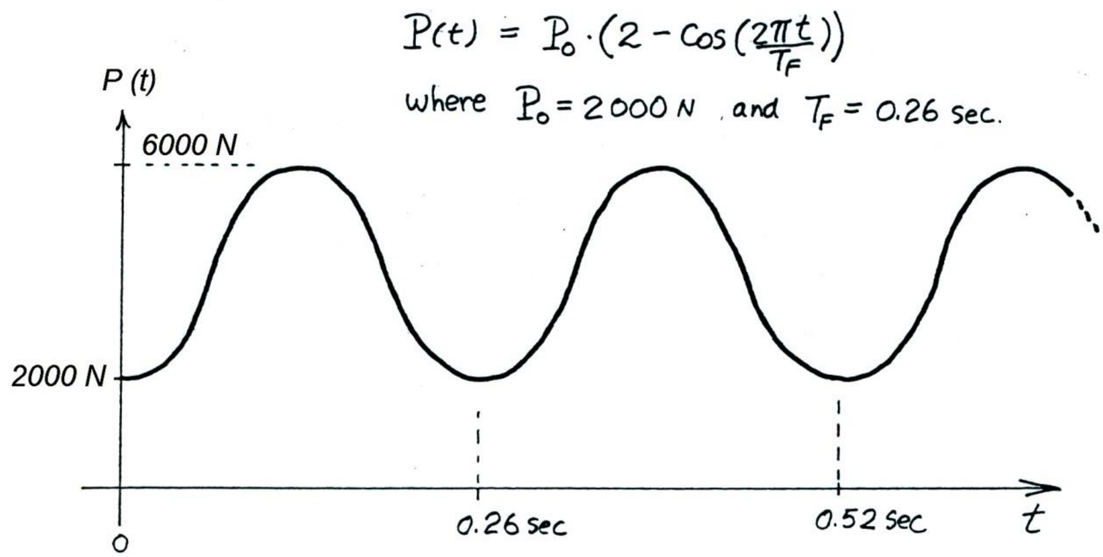
Important: For **non-periodic** functions, one can argue that they are periodic with an infinite period, that is, $L \to \infty$. The Fourier Series then becomes the **Fourier Integral**.

2.7. Solved Examples: Response of SDF Systems to Harmonic and Period Loading

Example 1: The SDF structure in Figure below is excited by a lateral dynamic force $P(t)$. The mass of the structure (m) is $4000\ kg$. The lateral stiffness of the structure (k) is $2 \times 10^6\ N/m$. The critical damping ratio (ξ) of the structure is 0.03.



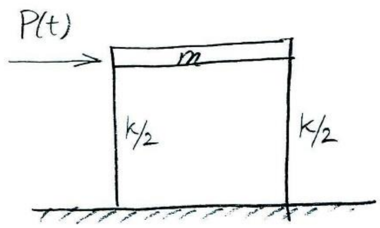
The force-time history of the dynamic force $P(t)$ is shown in Figure below.



Assuming that the structural response to $P(t)$ has already reached the steady state condition, determine the maximum lateral displacement of the SDF structure.

Note: The dynamic force $P(t) = P_o(2 - \cos(2\pi t/T_f)$ in the above figure can be treated as a superposition of a static force $2P_o$ and a harmonic force $P_o\cos(2\pi t/T_f)$

Solution:



$m = 4000\,kg$
$k = 2.0 \times 10^6\,N/m$
$\xi = 0.03$

$$P(t) = P_o(2 - \cos\omega_F t) = \underbrace{2P_o}_{\text{Static Force}} - \underbrace{P_o\cos\omega_F t}_{\text{Dynamic Force}}$$

Response to the static Force $2P_o$:

$$u_s = 2P_o/k = 2 \times 2000 / 2.0 \times 10^6\,N = 0.002\,m$$

Max. Response to the Dynamic Force:

$$u_{d,max} = \frac{P_o}{K} \cdot D$$

where $$D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

$$\beta = \omega_F / \omega_n$$

$$\omega_F = 2\pi / T_F = 2\pi / 0.26 = 24.166 \text{ rad/sec}$$

$$\omega_n = \sqrt{K/m} = \sqrt{2\times10^6 / 4000} = 22.36 \text{ rad/sec}$$

$$\omega_F / \omega_n = 1.0807$$

$$D = \frac{1}{\sqrt{(1-1.0807^2)^2 + (2\times0.03\times1.0807)^2}} = \underline{5.555}$$

$$\therefore \quad u_{d,max} = \frac{2000}{2\times10^6} \times 5.555 = 5.555\times10^{-3} \text{ m}$$

Max. Lateral Displacement u_{max} :

$$u_{max} = u_s + u_{d,max} = \underline{\underline{0.00756 \text{ m}}}$$

Example 2: For the same case of example 1, Now, suppose that it is necessary to reduce the maximum lateral displacement under this dynamic loading to approximately one-third of the above value, and there are two schemes to be considered for this purpose:

Scheme 1: The first scheme is to increase the lateral stiffness of the structure. This can be done by stiffening the columns. However, due to some practical limitations, the maximum stiffness, after the stiffening process, will not be higher than $1.2k$, where k is the original stiffness.

Scheme 2: The second scheme is to increase the structural mass. This can be done by attaching an additional mass on top of the structure. Again, this scheme has a limitation. The additional mass must not be greater than 1000 kg.

Which scheme will you choose? Why?

If you choose the first scheme, what is the minimum additional lateral stiffness of the structure required to achieve the target reduced response? If you choose the second scheme, what is the minimum additional mass required to achieve the target reduced response?

Solution:

Consider 2 schemes to reduce the max. lateral displacement:

#1 Increase the lateral stiffness.

#2 Increase the structural Mass.

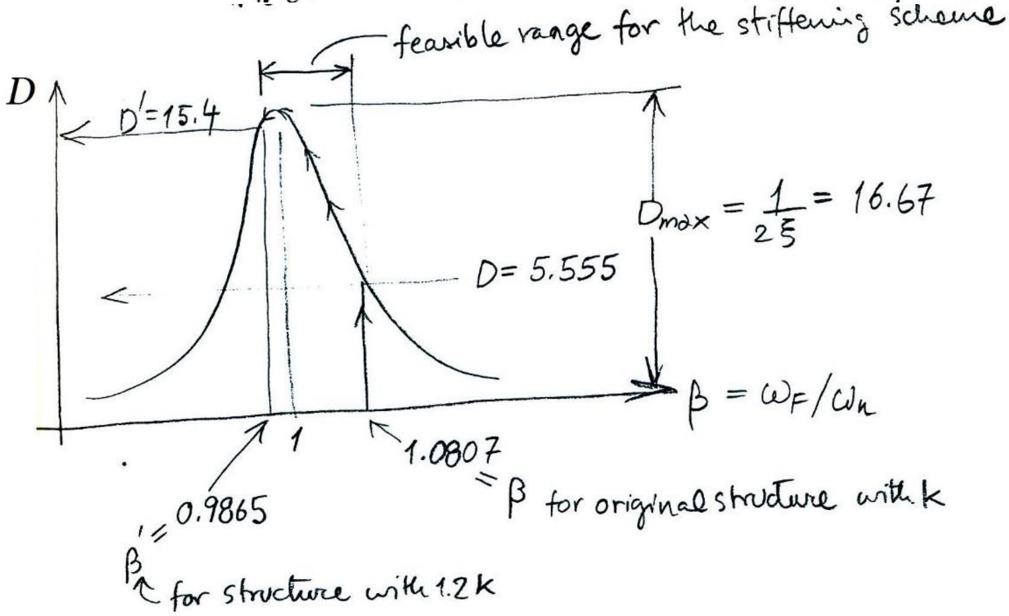
Scheme #1.

If the stiffness is increased to 1.2K, then

$$\omega_n' = \sqrt{\frac{1.2k}{m}} = \sqrt{\frac{1.2\times2\times10^6}{4000}} = 24.494 \text{ rad/s.}$$

$$\beta' = \omega_F/\omega_n' = 24.166/24.494 = 0.9865$$

$$D' = \frac{1}{\sqrt{(1-0.9865^2)^2+(2\times0.03\times0.9865)^2}} = 15.4$$



This scheme will not lead to a reduction in the lateral displacement because the increase in the lateral stiffness will result in a rapid increase in the Dynamic Magnification Factor.

Scheme #2

Adding Mass $\longrightarrow$ Reduce the natural frequency
$\downarrow$
Reduce the Dynamic Magnification Factor.
$\downarrow$
Reduce the lateral Response

The targeted max. lateral response $= \frac{1}{3} \times 0.00756\,m = 2.52\times10^{-3}\,m$

The static Response $U_S = 2.0\times10^{-3}\,m$

$\therefore$ the max. dynamic response $u'_{d,max}$ should be

$$u'_{d,max} = 2.52\times10^{-3} - 2.0\times10^{-3} = 0.52\times10^{-3}\,m$$

ccording to the theory,

$$u'_{d,max} = \frac{P_0}{K} \cdot D' = \frac{2000}{2\times10^6} \times D' = 1\times10^{-3} D'$$

Hence $D' = \frac{0.52\times10^{-3}}{1.0\times10^{-3}} = \underline{0.52}$

But $$D' = \frac{1}{\sqrt{(1-\beta'^2)^2 + (2\xi\beta')^2}}$$

$$0.52^2\left\{(1-\beta'^2)^2 + (2\xi\beta')^2\right\} = 1.0$$

$$(1-\beta'^2)^2 + (2\times 0.03\times\beta')^2 = 1.0/0.52^2 = 3.698$$

$$\beta' \cong 1.708$$

$$\therefore \quad \omega_n' = \omega_F/\beta' = 24.166/1.708 = 14.148 \text{ rad/sec}$$

$$\omega_n' = \sqrt{\frac{k}{m'}} \quad \therefore \quad m' = \frac{k}{\omega_n'^2} = \frac{2.0\times 10^6}{14.148^2}$$

$$\underline{m' = 9{,}990 \text{ kg}}$$

Required Additional Mass to achieve the targeted performance

$= 9900 - 4000 = 5{,}990$ kg > 1000 kg.

Try the max. additional mass of 1000 kg.

$$\omega_n' = \sqrt{\frac{2.0\times10^6}{4000+1000}} = 20.0 \text{ rad/sec}$$

$$\beta' = 24.166/20.0 = 1.2083$$

$$D' = 2.14746$$

$$u'_{d,max} = 1.0\times10^{-3}\times 2.14746 = 2.147\times10^{-3} \text{ m}$$

$$u'_{max} = 2.0\times10^{-3} + 2.147\times10^{-3} = 4.147\times10^{-3} \text{ m}$$

which is about 55 % of the original value.

Not as low as 33% (as required)

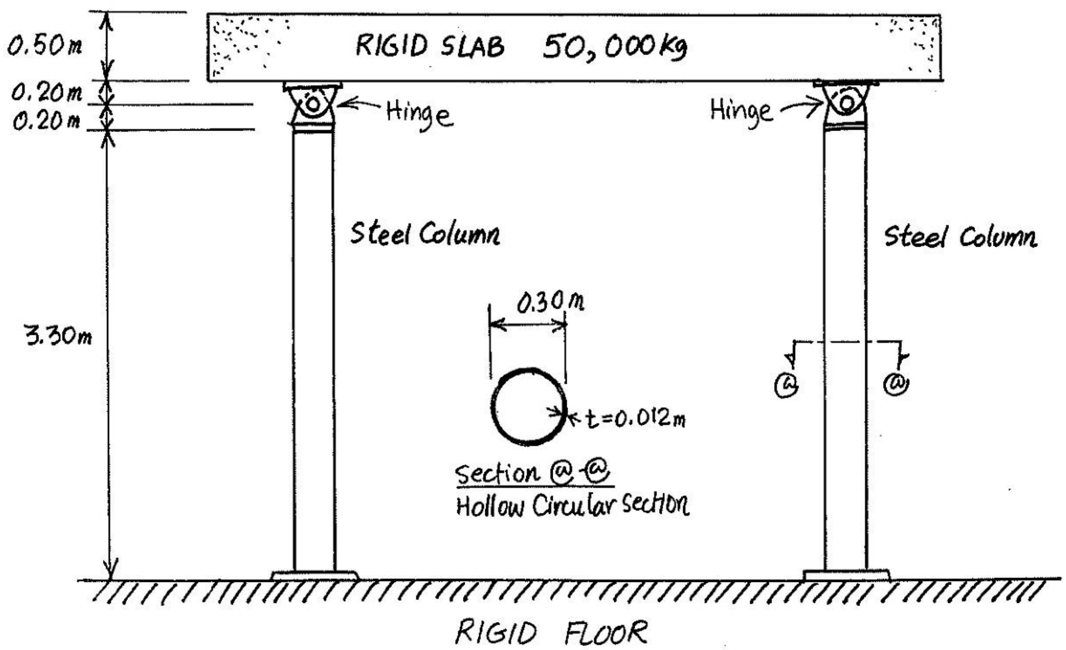
Though it is not possible to achieve the targeted performance, this scheme #2 is definitely better than the scheme #1.

So, we should select scheme #2.

Example 3: The structure shown in Figure below is composed of a rigid slab and two supporting steel columns. Each of these columns has a hollow circular section with external diameter of 0.300 m and internal diameter of 0.276 m. The connections between the top slab and the supporting columns are hinge supports, while the connections between the columns and the rigid floor are fixed end type. Important structural properties and dimensions are presented in Figure. The critical damping ratio of this structure is 0.03.

Compute the natural frequency of this structure. Note that the mass of the columns is very small when compared with the mass of the platforms, so the effects of column mass can be neglected.

Suppose that the structure is subjected to a lateral periodic ground displacement $u_g(t)$ shown in Figure.



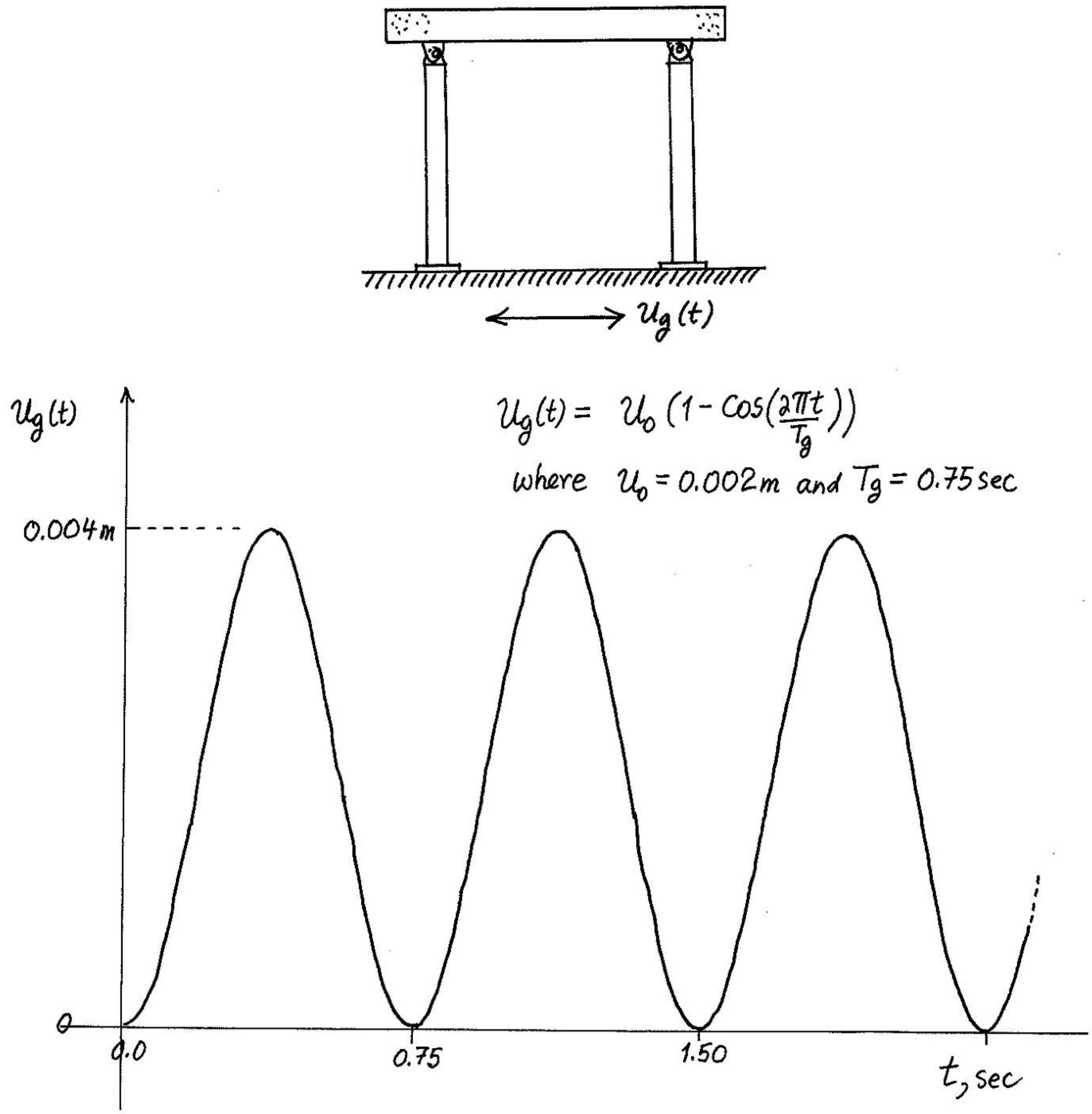
A rigid slab supported by two steel columns

Before the application of this ground displacement, the structure is at rest i.e. it has no initial displacement or motion. The response of interest for this loading case is the steady state response.
Find the maximum lateral displacement of the rigid slab relative to the ground.

Find the maximum shear force in each column.

Find the maximum bending stress in each column.

Note that it is not necessary to compute the "exact" values of these responses. Approximate values with a reasonable accuracy (say, within 5% error) are good enough.



Harmonic ground motion at the base of system

Now, suppose that the maximum displacement and bending stress in this loading case are too high, and it is necessary to reduce these maximum responses by at least 30%. Two schemes to reduce the responses are proposed:

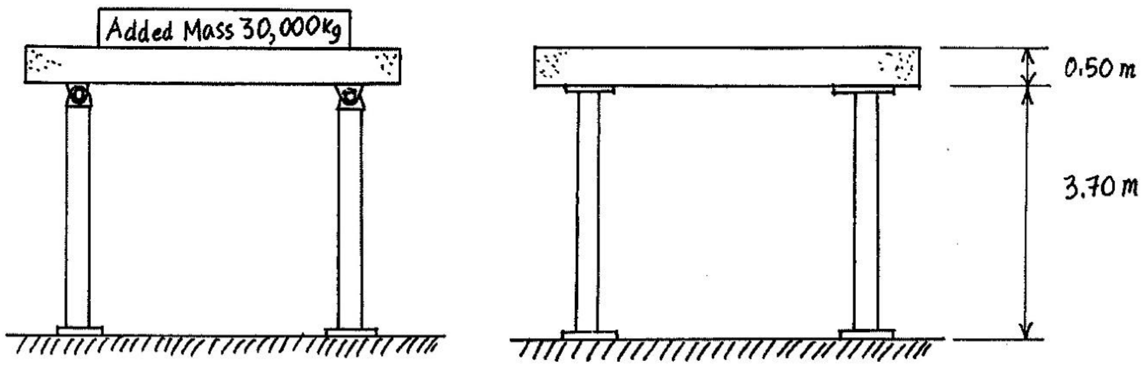
Scheme A: Add rigid mass of 30,000 kg to the top rigid slab as shown by Figure. The added mass is firmly locked to the slab. By this scheme, it may be assumed that the critical clamping ratio remains unchanged (equal to 0.03).

Scheme B: Replace the supporting columns by two new steel columns as shown by Figure. The cross-sectional properties of new columns are identical to the original ones [external diameter of 0.300 m and internal diameter of 0.276 in]. The connections between columns and rigid slab and rigid floor are fixed end type. By this scheme, it may be assumed that the critical damping ratio remains unchanged (equal to 0.03).

Determine the effectiveness of each of these two schemes.

Which scheme should be adopted?

What are the maximum displacement responses under the adopted scheme?

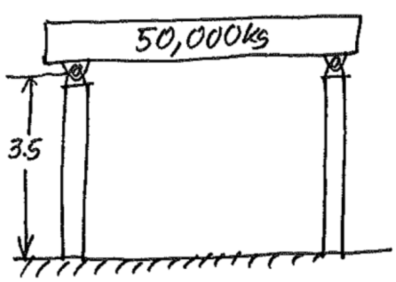


Scheme A

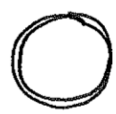
Scheme B

Proposed schemes to reduce the maximum responses

Solution:



Steel Column's Cross Sectional Properties



$$I_x = \frac{\pi}{64} \times (d_o^4 - d_i^4)$$

$$= \frac{\pi}{64} \times (0.30^4 - 0.276^4)$$

$$I_x = 1.1276 \times 10^{-4}\ m^4$$

$$E = 2.0 \times 10^{11}\ N/m^2$$

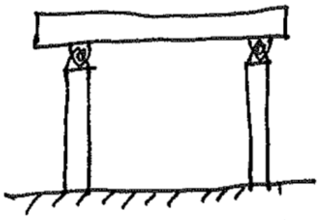
$$K_c = \frac{3EI_x}{L_c^3} = \frac{3 \times 2.0 \times 10^{11} \times 1.1276 \times 10^{-4}}{3.5^3} = 1.578 \times 10^6\ N/m$$

$$M_c = 50{,}000\,kg$$

The Natural frequency of the structure:

$$f_n = \frac{1}{2\pi} \cdot \sqrt{\frac{2K_c}{M_c}} = \frac{1}{2\pi} \times \sqrt{\frac{1.578 \times 10^6 \times 2}{50{,}000}} = \underline{1.2644\ HZ}$$

Loading Case 2



$$u_g(t) = u_0(1 - \cos(\omega_g t)), \quad \omega_g = 2\pi/T_g$$

$$\dot{u}_g(t) = 0 + \omega_g u_0 \sin\omega_g t$$

$$\ddot{u}_g(t) = \omega_g^2 u_0 \cos\omega_g t$$

$$\text{Effective Force} = -m\ddot{u}_g(t) = -\underbrace{m\omega_g^2 u_0}_{P_0}\cos\omega_g t$$

$T_g = 0.75$, $\omega_g = 2\pi/T_g = 8.3776$ rad/sec, $u_0 = 0.002$ m

$$P_0 = 50{,}000 \times 8.3776^2 \times 0.002 = \underline{7018.4\ N}$$

$$\beta = \omega_g/\omega_n = \frac{8.3776}{2\pi \times 1.2644} = \frac{8.3776}{7.9448} = \underline{1.0545}$$

$$D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} = \frac{1}{\sqrt{(1-1.0545^2)^2 + (2\times 0.03 \times 1.0545)^2}}$$

$$= \frac{1}{\sqrt{0.01252 + 0.004}} = \underline{7.78}$$

$$u_{max} = \frac{7018.4}{2k_c} \times D = \frac{7018.4}{2\times 1.578\times 10^6} \times 7.78$$

$\underline{\underline{u_{max} = 0.0173\ m}}$ Max. Lateral Displacement

$K_c u_{max} = 27301.6$ N

$M_{max} = 27301.6 \times 3.5 = 95{,}555.5$ N.m

$\sigma_{max} = 95{,}555.5 \times 0.15 / 1.1276\times 10^{-4} = \underline{\underline{127.1\times 10^6\ N/m^2}}$ Max. Stress

Consider Scheme A & B to reduce the displacement and stress responses.

Scheme A Add Mass

$$\omega_n \rightarrow \omega_n' = \sqrt{\frac{2\times 1.578\times 10^6}{80{,}000}} = 6.281 \text{ rad/sec.}$$

$$\beta \rightarrow \beta' = 8.3776/6.281 = 1.334$$

$$P_o \rightarrow P_o' = 80{,}000\times 8.3776^2\times 0.002 = 11{,}229.5 \text{ N}$$

$$D \rightarrow D' = \frac{1}{\sqrt{0.6077+0.0064}} = 1.276$$

$$u_{max}' = \frac{P_o'}{2K_c}\times D' = \frac{11{,}229.5}{2\times 1.578\times 10^6}\times 1.276 = \underline{\underline{4.54\times 10^{-3}\text{ m}}} < \underline{0.7\times 0.0173}$$

$$\sigma_{max}' = \frac{127.1\times 10^6}{0.0173}\times 4.54\times 10^{-3} = \underline{\underline{33.4\times 10^6 \text{ N/m}^2}} < \underline{0.7\times 127.\text{MPa}}$$

This scheme is effective!

Scheme B Change Columns

$$K_c \rightarrow K_B = 5.343\times 10^6 \text{ N/m}$$

$$\omega_n \rightarrow \omega_n'' = \sqrt{\frac{2\times 5.343\times 10^6}{50{,}000}} = 14.62 \text{ rad/sec}$$

$$\beta \rightarrow \beta'' = 8.3776/14.62 = 0.573$$

$$P_o \rightarrow P_o'' = 50{,}000\times 8.3776^2\times 0.002 = 7{,}018.4 \text{ N}$$

$$D \rightarrow D'' = \frac{1}{\sqrt{(1-0.573^2)^2+(2\times 0.03\times 0.573)^2}} = \frac{1}{\sqrt{0.451+0.0012}}$$

$$= \underline{1.487}$$

$$u''_{max} = \frac{P''_o}{2K_B} \cdot D'' = \frac{7018.4}{2 \times 5.343 \times 10^6} \times 1.487 = \underline{\underline{9.77 \times 10^{-4}\ m}}$$

$$\text{Max. Shear} = K_B u''_{max} = 5218\ N$$

$$\text{Max B.M} = 5218 \times 3.70/2 = 9653.6\ N.m$$

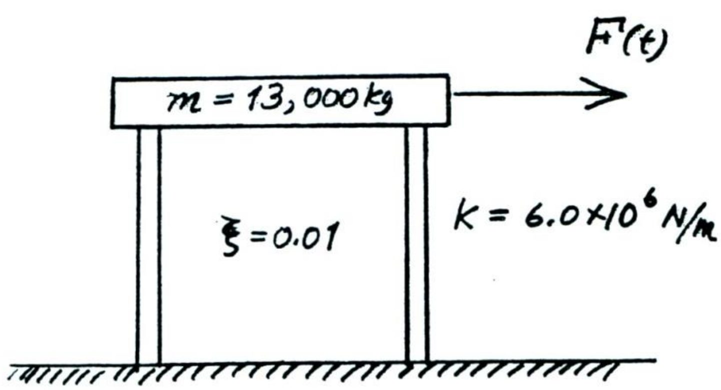
$$\sigma''_{max} = \frac{9653.6 \times 0.15}{1.1276 \times 10^{-4}} = \underline{\underline{12.8 \times 10^6\ N/m^2}}$$

This scheme B is also effective.

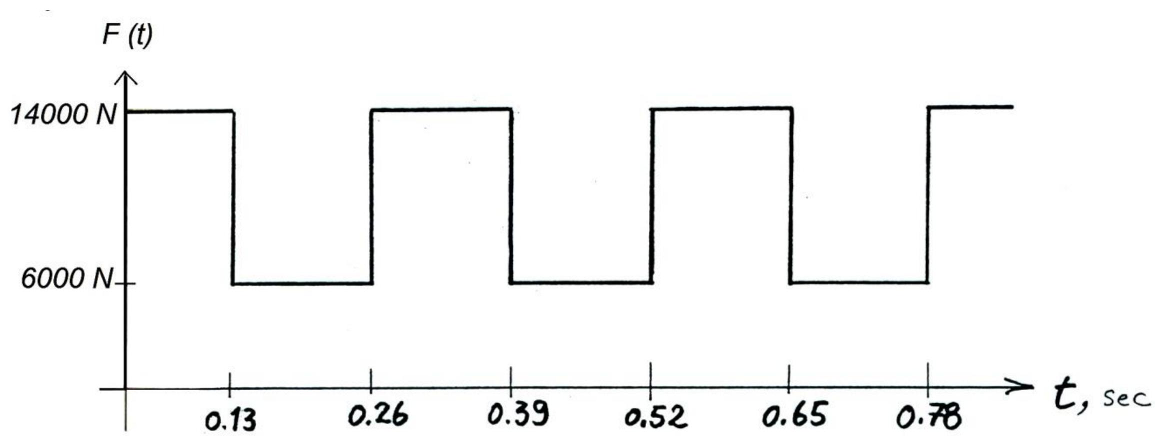
It is even more effective than scheme B.

Any of these two schemes can be adoped!

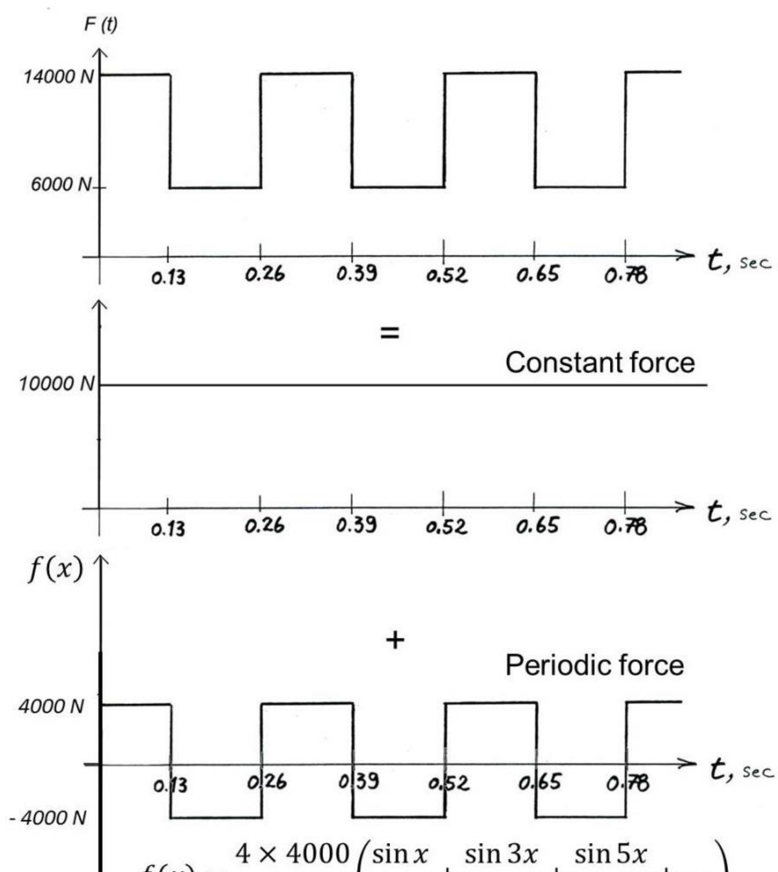
Example 4: A single story building shown in Figure below is subjected to a lateral periodic force $F(t)$. The building top mass (m) is 13,000 Kg. The combined lateral stiffness of supporting columns (k) is $6 \times 10^6\ N/m$. The critical damping ratio of the building is 0.01.



Assuming that the response has already reached the steady-state condition, find the maximum lateral displacement of the building.



Note: The dynamic force $F(t)$ in the above figure can be treated as a superposition of a static force of 10,000 N and a square periodic force function $f(x)$ as shown below ($x = \bar{\omega}t$).



$$f(x) = \frac{4 \times 4000}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Solution:

single-story building

$m = 13{,}000\,kg$

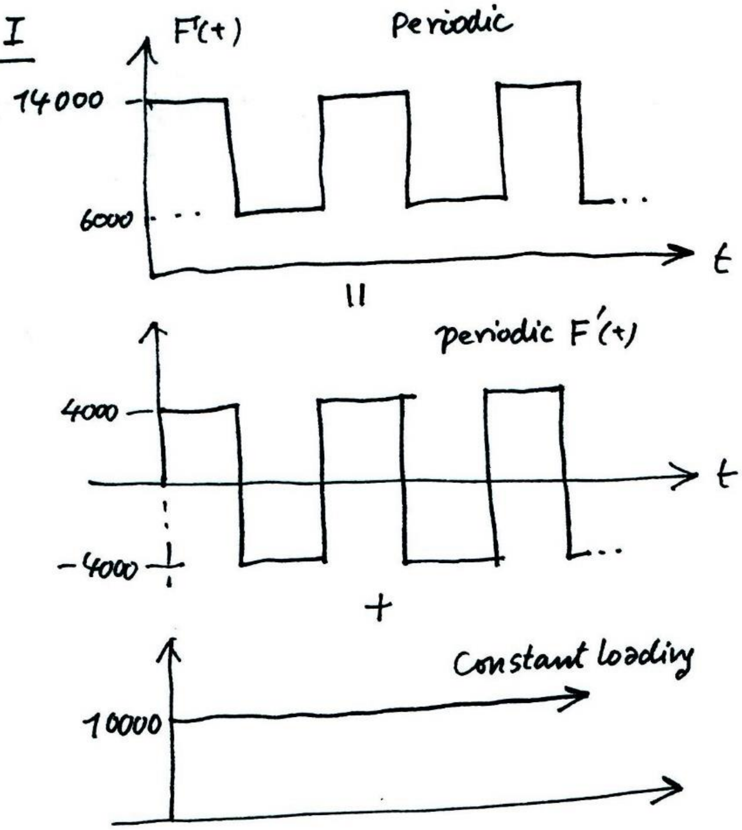
$k = 6\times10^{6}\,N/m$

$\omega = \sqrt{\frac{k}{m}} = 21.48$ rad/sec

$f = \omega/2\pi = 3.419$ HZ

$T = 1/f = 0.292$ sec. , $\xi = 0.01$

Case I



steady-state response to the constant loading

$= 10000\ N / 6\times10^{6}\ N/m = 1.667\times10^{-3}\ m$

Periodic loading $F'(t)$ Can be prepented by

$$F'(t) = \sum_{n=1}^{\infty} b_n \sin(n\bar{\omega}t)$$

Where $b_1 = 4\times4000/\pi \cong 5093\ N$

$b_2 = 0$

$b_3 = 4\times4000/3\pi \cong 1698\ N.$

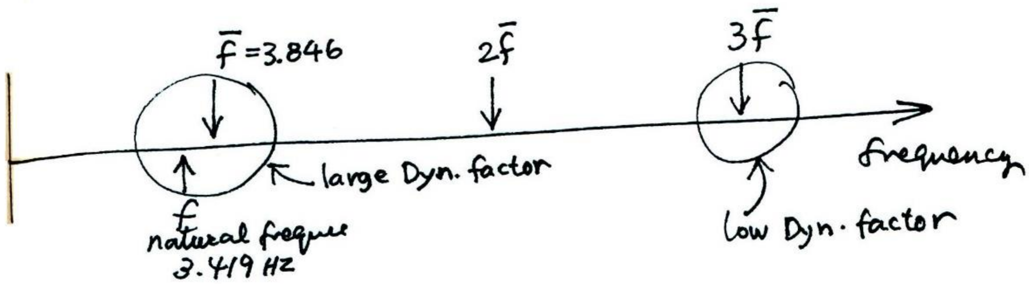
$b_4 = 0$

$\vdots$

period of periodic force $\bar{T} = 0.26$ sec

$\bar{f} = 1/\bar{T} = 3.846\ Hz$

$\bar{\omega} = 2\pi f = 24.17$ rad/sec



Amplitude to the first harmonic

$$a_1 = \frac{b_1}{k} \cdot D_1$$

$$D_1 = \frac{1}{\sqrt{(1-\beta_1^2)^2 + (2\xi\beta_1)^2}} \quad ; \quad \beta_1 = \bar{f}/f = 1.1248$$

$D_1 = 3.756$

$$a_1 = \frac{5093}{6\times10^{6}} \times 3.756 = 3.1896\times10^{-3}\ m$$

Amplitude to the third harmonic

$$a_3 = \frac{b_3}{k} \cdot D_3$$

$$D_3 = \frac{1}{\sqrt{(1-\beta_3^2)^2 + (2\xi\beta_3)^2}} \quad ; \quad \beta_3 = 3\bar{f}/f = 3.3744$$

$$D_3 = 0.0963 \longleftarrow \text{very low!}$$

$$a_3 = \frac{1698}{6\times10^6} \times 0.0963 = \underline{2.724\times10^{-5}\ m}$$

negligible when compared with a_1

the maximum lateral displacement to the loading Case I is

$$\underbrace{1.667\times10^{-3}}_{\text{caused by constant loading}} + 3.1896\times10^{-3} = \underline{4.857\times10^{-3}\ m}$$

Ans

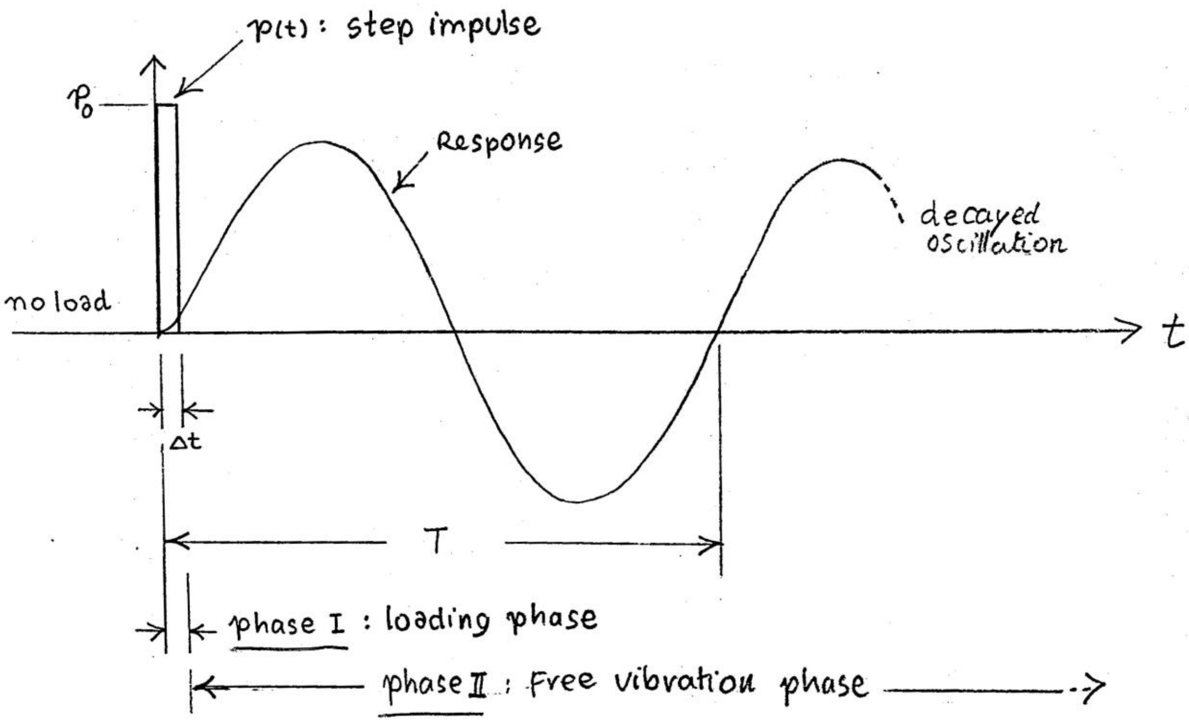
2.8. Response to Impulse Loading

Impulsive Shock loads, short duration loads.

Impact, blast wave, explosion

truck/auto mobiles/travelling cranes

The study of impulse response is also important for the analysis of response to arbitrary loadings.



Impulse force: magnitude = p_o, start at $t = 0$

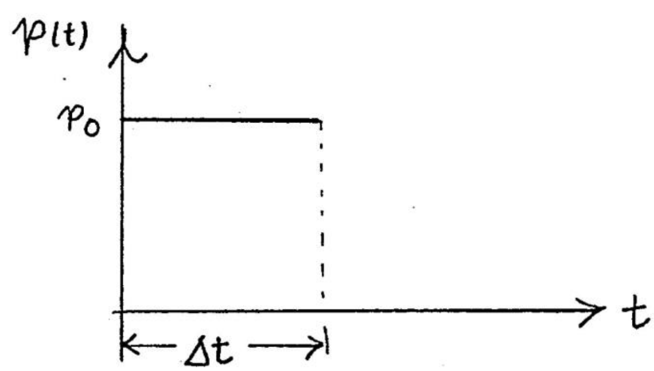
Duration = Δt, where $\Delta t/T \ << 1$

Structure: initial at-rest $u(0) = 0,\ \dot{u}(0) = 0$

Phase 1:

The particular to a step loading is simply a static deflection:

$$u_p(t) = p_o/k$$



This solution satisfies the equation of equilibrium.

$$\dot{u}_p(t) = 0$$

$$\ddot{u}_p(t) = 0$$

Putting in equation of motion,

$$0 + 0 + k\frac{p_0}{k} = p_0$$

A general solution

$$u(t) = u_p(t) \; + \; u_h(t)$$

$$u(t) = \frac{p_o}{k} + e^{-\xi\omega t}[A\sin(\omega_D t) + B\cos(\omega_D t)]$$

Where

A and B are determined such that at-rest initial conditions are satisfied.

$$u(0) = \frac{p_o}{k} + B = 0$$

$$B = -\frac{p_o}{k}$$

$$\dot{u}(0) = \omega_D A - \xi\omega B = 0$$

$$A = \frac{\xi\omega B}{\omega_D}$$

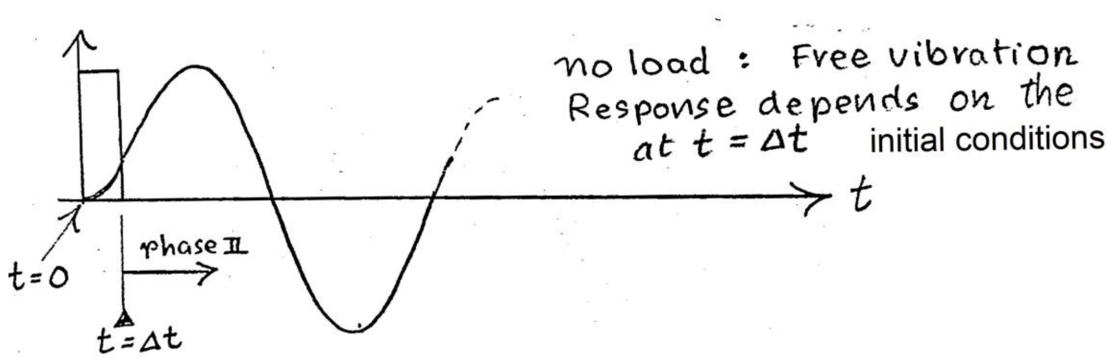
So we obtain (in the range $0 < t < \Delta t$)

$$u(t) = \frac{p_o}{k} + e^{-\xi\omega t}\left[-\frac{\xi\omega B}{\omega_D}\frac{p_o}{k}\sin(\omega_D t) - \frac{p_o}{k}\cos(\omega_D t)\right]$$

Next, due to fact that $\Delta t/T \; << 1$ and $\xi << 1$

$$u(t) \approx \frac{p_o}{k} - \frac{p_o}{k}\cos(\omega t) \approx \; \frac{p_o}{k}(1 - \cos(\omega t))$$

Phase 2:



$$u(t) = \frac{p_o}{k}(1 - \cos(\omega t)), \qquad u(\Delta t) = \frac{p_o}{k}(1 - \cos(\omega \Delta t))$$

$$\dot{u}(t) = \frac{p_o}{k}(\omega \sin(\omega t)), \qquad \dot{u}(\Delta t) = \frac{p_o}{k}(\sin(\omega \Delta t))$$

Employing Tylor's expansion:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots \dots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots \dots$$

For small θ, by neglecting the second order term and higher terms, we get

$$\sin\theta \approx \theta, \qquad \cos\theta \approx 1$$

Introducing this approximation in above equations, we obtain

$$u(t) = 0$$

$$\dot{u}(t) \cong \frac{p_o \omega^2 \Delta t}{k} = \frac{p_o \Delta t}{m}$$

$p_o \Delta t$ is an impulse.

The above equation says that impulse ≈ the change in momentum (of the mass)

The impulse introduces "momentum" into a structure but the duration of impulse is so short that the displacement has not been developed yet.

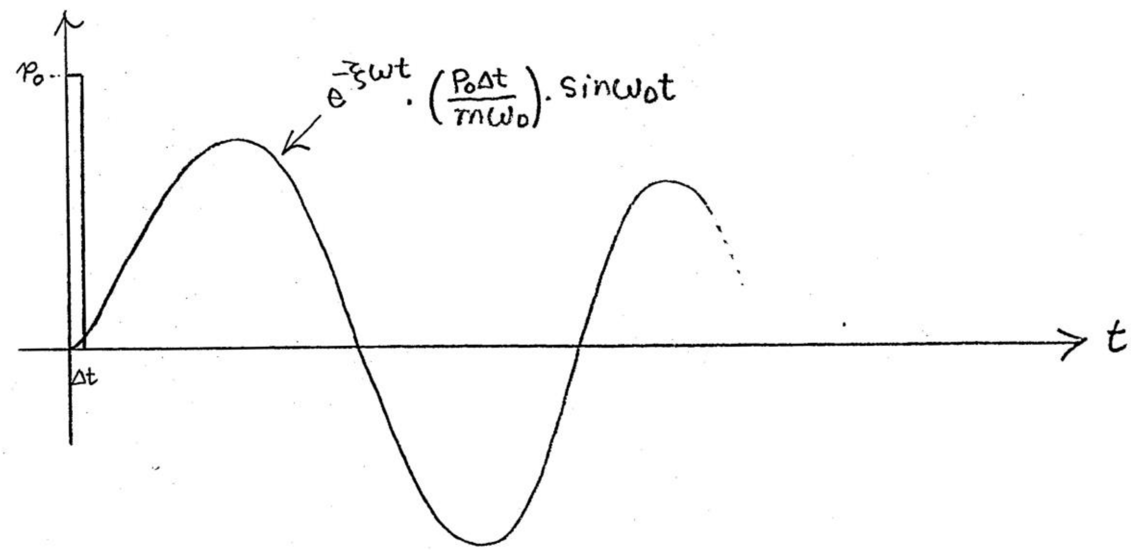
Using the above two equations as the initial conditions for free vibration in Phase 2,

$$u(t) = e^{-\xi\omega t}\left[\frac{\dot{u}(\Delta t)}{\omega_D} \sin\omega_D(t - \Delta t)\right]$$

Since $\Delta t/t \ll 1$ it is justified to let $t - \Delta t \approx t$ that is

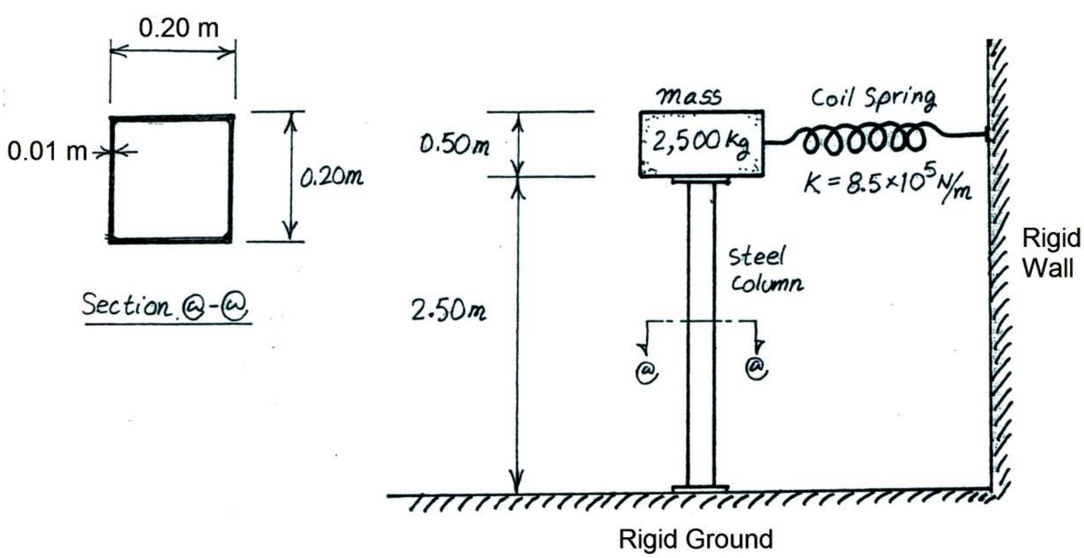
$$u(t) = e^{-\xi\omega t}\frac{p_o \Delta t}{m\omega_D} \sin(\omega_D t)$$

The above equation will be used when we analyze the response to arbitrary loading in the next section.

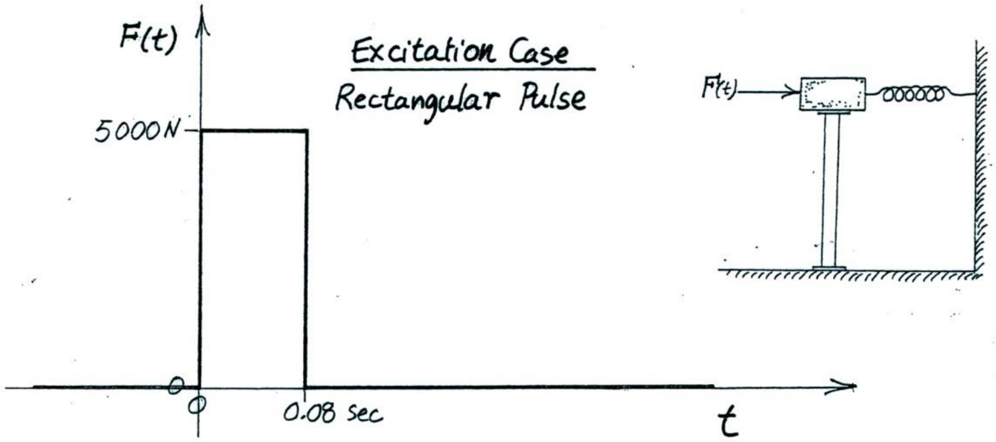


2.9. Solved Examples: Response to Impulse Loading

Question 1: The structure in the following Figure is composed of a top lumped mass, a supporting steel column, and a lateral coil spring. The column has a hollow square section as shown in the Figure. The mass, stiffness of the coil spring, and other important structural properties, as well as key dimensions, are all presented in Figure. The critical damping ratio of the structure is 0.03.
(a) Find the natural frequency and natural period of this structure.



Suppose that the structure is subjected to a lateral force F(t). The time history of $F(t)$ is shown in the following Figure.



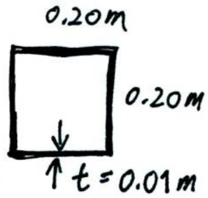
Before it was subjected to the excitation, the structure was in the rest condition.

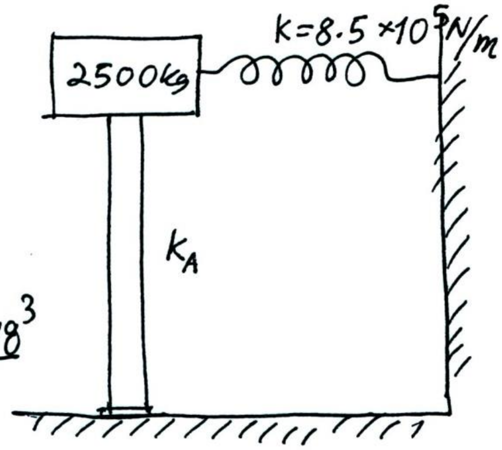
(b) Find the maximum lateral displacement of the top mass.

(c) Find the maximum bending moment and shear at the column base.

(d) Find the maximum stress in the column.

(e) Find the maximum force in the coil spring.





$$I = \frac{0.20\times0.20^3}{12} - \frac{0.18\times0.18^3}{12}$$

$$I = 4.585\times10^{-5}\ m^4$$

$$k_A = \frac{3\times2\times10^{11}\times4.585\times10^{-5}}{(2.50+0.25)^3} = 1.323\times10^{6}\ N/m$$

$$k_T = k_A + k = 1.323\times10^{6} + 8.5\times10^{5} = 2.173\times10^{6}\ N/m$$

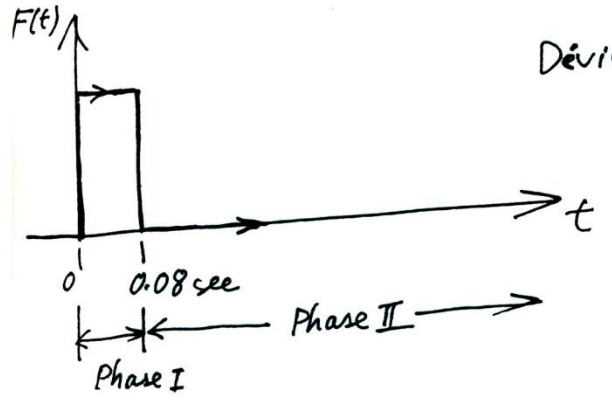
$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_T}{m}} = \frac{1}{2\pi}\sqrt{\frac{2.173\times10^{6}}{2500}} = 4.692\ HZ$$

$(\omega_n = 29.482\ rad/sec)$

$$T_n = 1/4.692 = 0.213\ sec$$

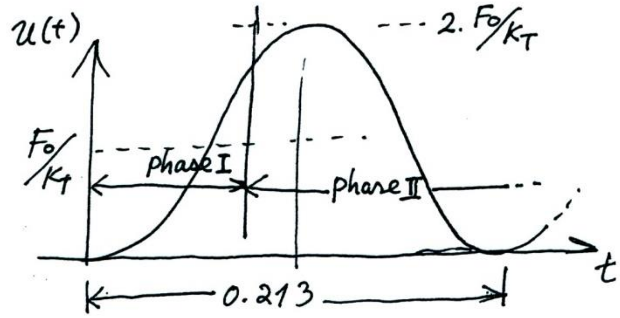
$T_0 = 0.08$ sec (Loading Case I) $\quad T_0 < T_n \quad$ but $T_0 \not\ll T_n$

$\therefore$ The loading Case I cannot be treated as an impulse.



Deviding the response into 2 phases:
Phase I: Loading Phase
Phase II: Free Vibration Phase

Phase I $\quad u(t) \cong \frac{F_o}{K_T}(1-\cos\omega_n t)$ (neglect the effect of damping)



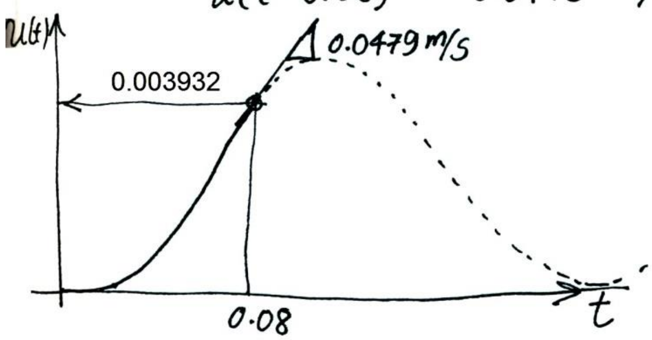
At the end of Phase I

$$u(t=0.08) = \frac{5000}{2.173\times10^6}\underbrace{\left(1-\cos(29.482\times0.08)\right)}_{1.7087}$$

$$= 3.932\times10^{-3}\ m.$$

$$\dot{u}(t) = \frac{du}{dt} = \frac{F_o}{K_T}\cdot\omega_n \sin(\omega_n t)$$

$$\dot{u}(t=0.08) = 0.0479\ m/s$$



Phase II $\quad$ Initial Disp $= 0.00392\ m = u_o$

Initial Vel $= 0.0479\ m/s = V_o$

Amplitude of the Oscillation $= \rho$

$$\rho = \sqrt{(0.00392)^2 + \left(\frac{V_o}{\omega_n}\right)^2} = \underline{0.00424\ m} = u_{max}$$

If the effect of damping is taken into account, the amplitude will be slightly lower:

Exact Results (including damping)

$u_0 = 0.00377\ m$

$V_0 = 0.0447\ m/s$

u_{max} (in phase 2) $= \underline{0.00406\ m}$

which is 4% lower than the approx. solution (0% damp)

Now, let's go back to the approx. solution:

Max. Lateral Displacement at the top mass $= \underline{0.00424\ m}$

Max. Shear at base $= V_{max} = K_A \times u_{max} = 1.323\times10^6 \times 0.00424$

$\underline{V_{max} = 5610\ N.}$

Max Moment at column base $= F_{max} \cdot h = V_{max} \times 2.75$

$\underline{M_{max} = 15428\ N.m}$

Max. Stress in Column $= \sigma_{max} = \dfrac{M_{max} \cdot C}{I}$

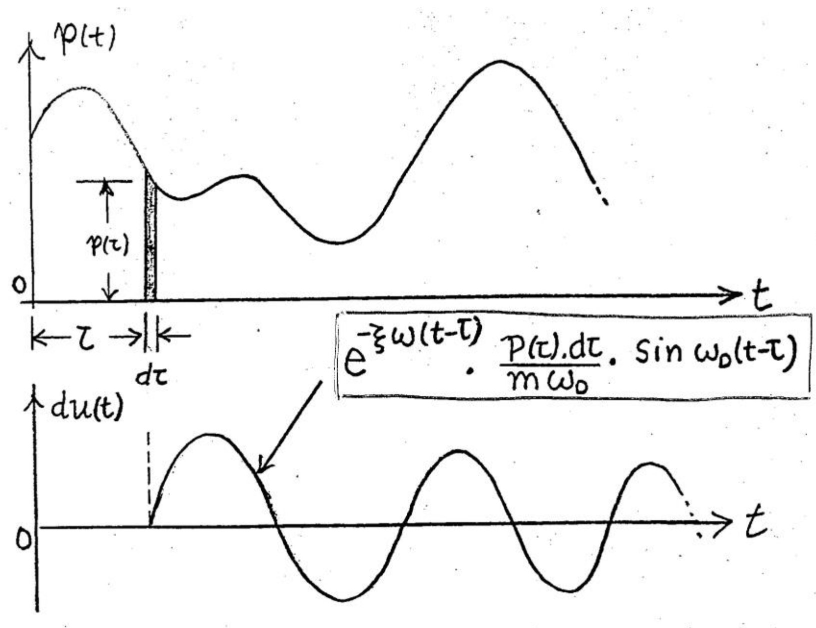
$\sigma_{max} = \dfrac{15428 \times 0.10}{4.585\times10^{-5}} = \underline{3.365\times10^{7}\ N/m^2}$

Max. Force in Coil Spring $= K_{spring} \cdot u_{max} = 8.5\times10^5 \times 0.00424$

$\underline{F_{max} = 3604\ N}$

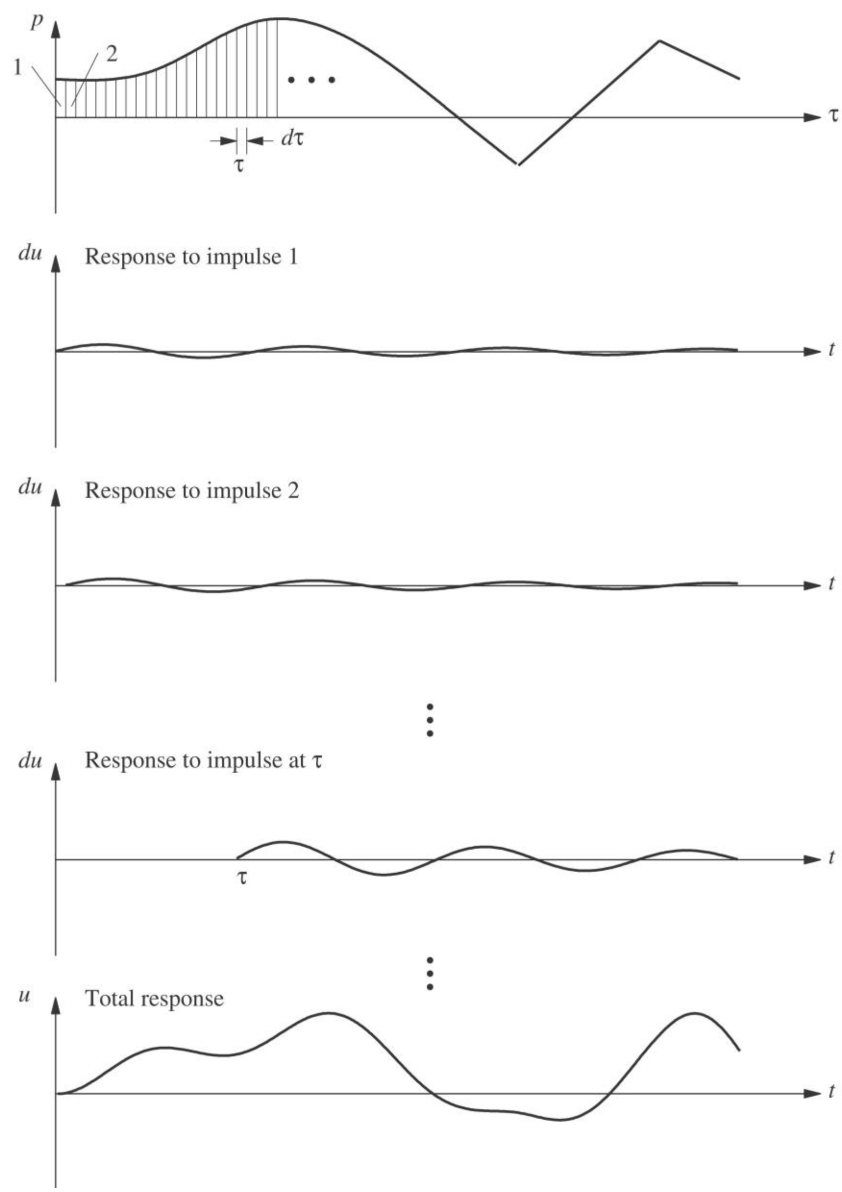
2.10. Response to General Dynamic Loading

2.10.1. Duhamel's Integral (Convolution Integral)



- A general dynamic, loading = A series of short Impulses
- Each impulse produce its own response
- The sum of these responses = the response to the dynamic loading

A force $p(t)$ varying arbitrarily with time can be represented as a sequence of infinitesimally short impulses.



Let $du(t;\tau)$ is the response of a linear dynamic system at time t due to impulse $p(\tau)d\tau$ at time τ.

$$du(t;\tau) = p(\tau)d\tau\, h(t-\tau)$$

Where

$$h(t-\tau) = \begin{cases} \dfrac{e^{-\xi\omega(t-\tau)}}{m\omega_D} \sin\omega_D(t-\tau), & t > \tau \\ 0, & t \le \tau \end{cases}$$

$h(t-\tau) =$ unit impulse response (or response to unit impulse applied at $t = \tau$).

By means of superposition the total responsive $u(t)$ can be obtained by summing all impulse responses developed during the loading history.

$$u(t) = \int_0^t p(\tau)\, h(t-\tau) d\tau$$

The integration is called “Convolution Integral” in general theory of mathematics.

Putting the unit impulse response yields the “Duhamel Integral” in structural dynamics.

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau)\, e^{-\xi\omega(t-\tau)}\, \sin\omega_D(t-\tau)\, d\tau$$

For an undamped system, Duhamel integral becomes,

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau)\, \sin\omega_D(t-\tau)\, d\tau$$

In above equation, it is assumed that the structure is initially at-rest condition i.e. $u(0) = 0,\ \dot{u}(0) = 0$.

For other cases, additional free vibration response must be added to the solution.

$$u(t) = e^{-\xi\omega t}\left[\frac{\dot{u}(0) + u(0)\,\xi\omega}{\omega_D} \sin\omega_D t + u(0)\cos\omega_D t\right] + \int_0^t p(\tau)\, h(t-\tau) d\tau$$

In the following investigation, the initial at-rest condition is assumed.

Duhamel’s integral provides a general result for evaluating the response of a linear SDF system to arbitrary force. This result is restricted to linear systems because it is based on the principle of superposition. Thus it does not apply to structures deforming beyond their linearly elastic limit. If $p(\tau)$ is a simple function, closed-form evaluation of the integral is possible and Duhamel’s integral is an alternative to the classical method for solving differential equations. If $p(\tau)$ is a complicated function that is described numerically, evaluation of the integral requires numerical methods.

Using the trigonometric identity, the integral equation of response (at-rest initial condition) can be expanded to the following.

$$\sin(\omega_D t - \omega_D \tau) = \sin(\omega_D t)\cos(\omega_D \tau) - \cos(\omega_D t)\sin(\omega_D \tau)$$

$$u(t) = \frac{1}{m\omega_D} \int_0^t p(\tau)\, e^{-\xi\omega(t-\tau)}\, [\sin(\omega_D t)\cos(\omega_D \tau) - \cos(\omega_D t) - \sin(\omega_D \tau)] d\tau$$

$$u(t) = \left[\frac{1}{m\omega_D} \int_0^t p(\tau)\, e^{-\xi\omega(t-\tau)} \cos(\omega_D \tau)\, d\tau\right] \sin(\omega_D t) - \left[\frac{1}{m\omega_D} \int_0^t p(\tau)\, e^{-\xi\omega(t-\tau)} \sin(\omega_D \tau)\, d\tau\right] \cos(\omega_D \tau)$$

$$u(t) = A(t)\sin\omega_D t - B(t)\cos\omega_D t$$

Where

$$A(t) = \frac{1}{m\omega_D}\left(\int_0^t p(\tau)\ \frac{e^{\xi\omega\tau}}{e^{\xi\omega t}}\ \cos\omega_D\tau\ d\tau\right)$$

$$B(t) = \frac{1}{m\omega_D}\left(\int_0^t p(t)\ \frac{e^{\xi\omega\tau}}{e^{\xi\omega t}}\ \sin\omega_D\tau\ d\tau\right)$$

For undamped case,

$$A(t) = \frac{1}{m\omega}\left(\int_0^t p(\tau)\cos\omega\tau\ d\tau\right)$$

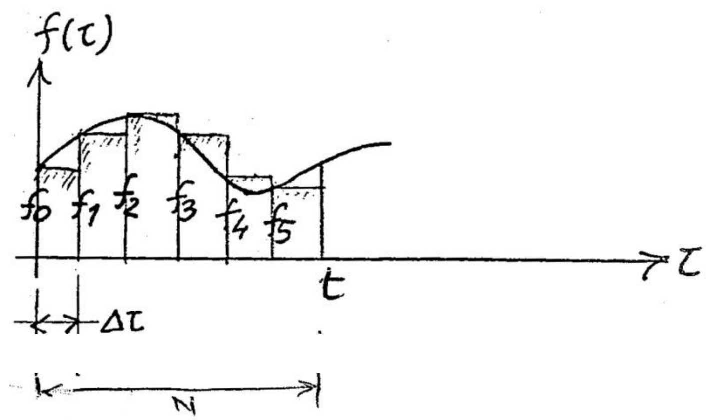
$$B(t) = \frac{1}{m\omega}\left(\int_0^t p(t)\sin\omega\tau\ d\tau\right)$$

The terms in brackets in above equations need numerical integration.

Simple Summation,

$$\int_0^t f(\tau)d\,\tau \cong \Delta\tau\ (f_0 + f_1 + f_2 + f_3 + \cdots f_{N-1})$$

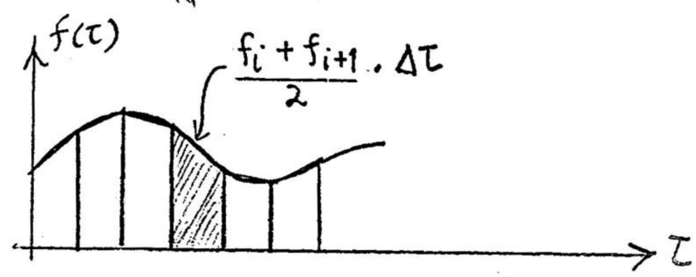
Where $f_i = f(i, \Delta\tau)$, $\Delta\tau = t/N$



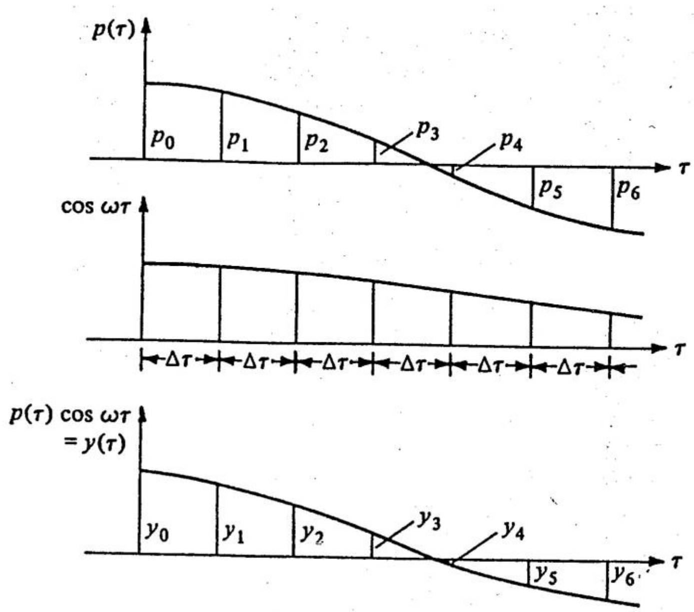
Or by Trapezoidal rule,

$$\int_0^t f(\tau)d\,\tau \cong \frac{\Delta\tau}{2}\ (f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots 2f_{N-1} + f_N)$$

Where $f_i = f(i, \Delta\tau)$, $\Delta\tau = t/N$



Consider first the numerical integration of $y(\tau) = p(\tau)\cos\omega\tau$ as required to find $A(t)$. For convenience of numerical calculation, the function $y(\tau)$ is evaluated at equal time increments $\Delta\tau$ as shown in Figure below, with the successive ordinates being identified by appropriate subscripts. The integral A_N can now be obtained approximately by summing these ordinates, after multiplying by weighting actors that depend on the numerical integration scheme being used as follows:



Formulation of numerical summation process for Duhamel integral

Simple summation:

$$A_N = \frac{\Delta\tau}{m\omega}[y_0 + y_1 + y_2 + \cdots + y_{N-1}]$$

Trapezoidal rule:

$$A_N = \frac{\Delta\tau}{2m\omega}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{N-1} + y_N]$$

Simpson's rule:

$$A_N = \frac{\Delta\tau}{3m\omega}[y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-1} + y_N]$$

Using any one of these equations, A N can be obtained directly for any specific value of N indicated. However, usually the entire time-history of response is required so that one must evaluate A_N for successive values of N until the desired time-history of response is obtained. For this purpose, it is more efficient to use these equations in their recursive forms:

Simple summation:

$$A_N = A_{N-1} + \frac{\Delta\tau}{m\omega}[y_{N-1}], \;\; N = 1, 2, 3, \ldots$$

Trapezoidal rule:

$$A_N = A_{N-1} + \frac{\Delta\tau}{2m\omega}[y_{N-1} + y_N], \;\; N = 1, 2, 3, \ldots$$

Simpson's rule:

$$A_N = A_{N-1} + \frac{\Delta\tau}{3m\omega}[y_{N-2} + 4y_{N-1} + y_N], \;\; N = 2, 4, 6, \ldots$$

Evaluation of $B(t)$ in can be carried out in the same manner, leading to expressions for B_N having exactly the same forms shown by above equation; however, in doing so, the definition of $y(\tau)$ must be changed to $y(\tau) = p(\tau)\sin\omega\tau$ consistent with the equation. Having calculated the values of A_N and B_N for successive values of N, the corresponding values of response u_N are obtained using

$$u_N = A_N \sin\omega t_N - B_N \cos\omega t_N$$

For critically damped systems,

Simple summation:

$$A_N = e^{-\xi\omega\Delta\tau} A_{N-1} + \frac{\Delta\tau}{m\omega_D}[y_{N-1}]e^{-\xi\omega\Delta\tau}, \;\; N = 1, 2, 3, \ldots$$

Trapezoidal rule:

$$A_N = e^{-\xi\omega\Delta\tau} A_{N-1} + \frac{\Delta\tau}{2m\omega_D}\left[y_{N-1}e^{-\xi\omega\Delta\tau} + y_N\right], \;\; N = 1, 2, 3, \ldots$$

Simpson's rule:

$$A_N = e^{-2\xi\omega\Delta\tau} A_{N-1} + \frac{\Delta\tau}{3m\omega_D}\left[y_{N-2}e^{-2\xi\omega\Delta\tau} + 4y_{N-1}e^{-\xi\omega\Delta\tau} + y_N\right], \;\; N = 2, 4, 6, \ldots$$

The expressions for B_N are identical in form to those given for A_N. Having calculated the values of A_N and B_N for successive values of N, the corresponding ordinates of response are obtained using

$$u_N = A_N \sin\omega_D t_N - B_N \cos\omega_D t_N$$

The accuracy to be expected from any of the above numerical procedures depends, of course, on the duration of time interval $\Delta\tau$. In general, this duration must be selected short enough for both the load and the trigonometric functions used in the analysis to be well defined, and further, to provide the normal engineering accuracy, it should also satisfy the condition $\Delta\tau \leq T/10$. Clearly the accuracy and computational effort increase with the complexity of the numerical integration procedure.

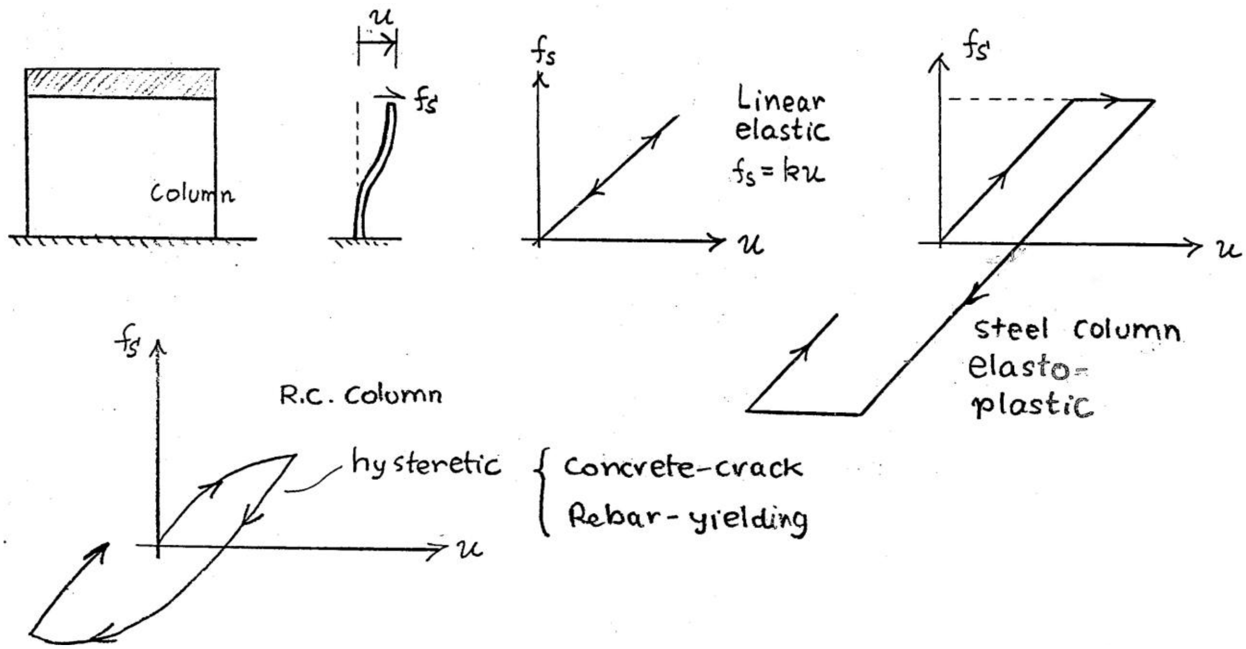
The concept of convolution integral will be used again later when we study the response of structures to random loadings from statistical view point (random vibration theory).

Note: The Convolution Integral is derived based on the principle of superposition. So, it is applicable only for the response analysis of "linear systems".

2.10.2. Step-by-step Numerical Integration Procedures (Time-stepping methods)

- General Dynamic loadings
- Nonlinear Structures

In some important structural dynamic problems, the responses of structures are in nonlinear range. For example, the response of structures subjected to a major earthquake.



For nonlinear analysis, Duhamel integral is not applicable. For such cases, we have “step-by- step direct integration procedures”.

Consider the dynamic equilibrium of a nonlinear structure at time t:

$$f_I(t) + f_D(t) + f_s(t) = p(t)$$

Where

$$f_I(t) = m\ddot{u}(t)$$

$$f_D(\dot{u}) \neq c\dot{u}(t)$$

Damping force is a nonlinear function of velocity.

$$f_s(u) \neq ku(t)$$

Restoring force is a nonlinear function of displacement.

At a same time Δt later:

$$f_I(t+\Delta t) + f_D(t+\Delta t) + f_s(t+\Delta t) = p(t+\Delta t)$$

Subtracting the original equation from this equation will yield the incremental form of differential equation of motion, as follows.

$$\Delta f_I(t) + \Delta f_D(t) + \Delta f_s(t) = \Delta p(t$$

Where

$$\Delta f_I(t) = f_I(t + \Delta t) - f_I(t) = m\,\Delta\ddot{u}(t)$$

$$\Delta f_D(t) = f_D(t + \Delta t) - f_D(t) \approx \left(\frac{df_D}{d\dot{u}}\right)_t \Delta\dot{u} = c(t)\Delta\dot{u}$$

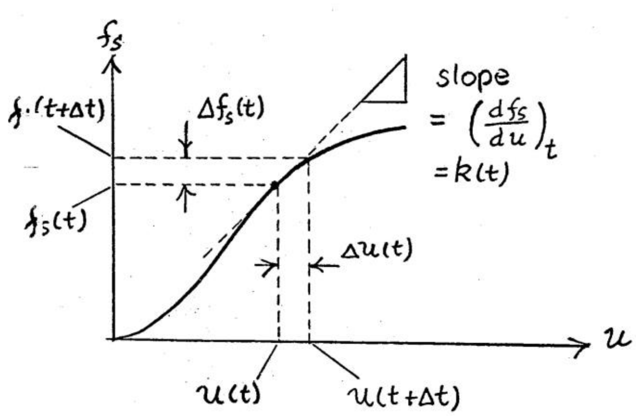
$$\Delta f_s(t) = f_s(t + \Delta t) - f_s(t) \approx \left(\frac{df_s}{d\dot{u}}\right)_t \Delta u = k(t)\Delta u$$

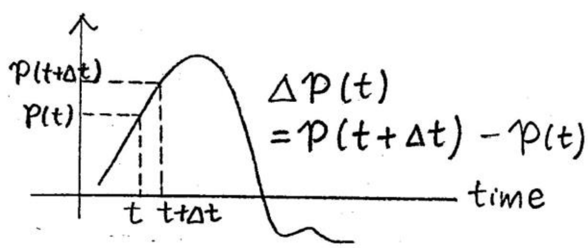
The later two equations are approximate. They can be used only if the change $\Delta\dot{u}$ and Δu are very small.

We have introduced the following two approximations.

$$\Delta f_D(t) = c(t)\Delta\dot{u}$$
$$\Delta f_s(t) = k(t)\,\Delta u$$

They are equivalent to the assumption that the damping force and restoring force are linear within t and $t + \Delta t$.





The incremental equation of motion becomes,

$$m\,\Delta\ddot{u}(t) + c(t)\Delta\dot{u} + k(t)\Delta u = \Delta p(t)$$

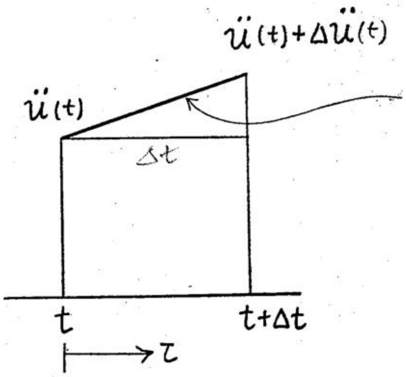
Stepping from time t to $t + \Delta t$ is usually not an exact procedure. Many approximate procedures are possible that are implemented numerically. The three important requirements for a numerical procedure are (1) convergence—as the time step decreases, the numerical solution should approach the exact solution, (2) stability—the numerical solution should be stable in the presence of numerical round-off errors, and (3) accuracy—the numerical procedure should provide results that are close enough to the exact solution.

There are three common types of time-stepping procedures.

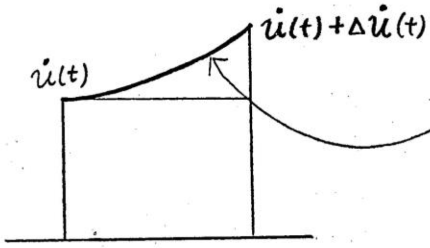
1) Methods based on interpolation of the excitation function
2) Methods based on finite difference expressions of velocity and acceleration (e.g. Central Difference Method), and
3) Methods based on assumed variation of acceleration (e.g. Newmark's Average Acceleration Method and Newmark's Linear Acceleration Method).

Here we will discuss the Newmark's Linear Acceleration Method.

Let's introduce an assumption that "the acceleration response varies linearly during each time increment". This yields quadratic and cubic variations of velocity and displacement, respectively.

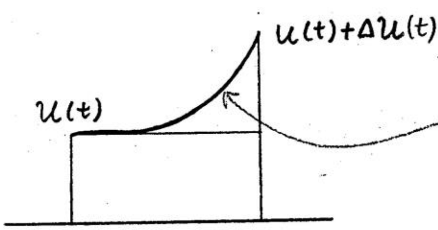


$$\ddot{u}(\tau) = \ddot{u}(t) + \frac{\Delta\ddot{u}}{\Delta t}.\tau$$



$$\dot{u}(\tau) = \dot{u}(t) + \int_0^\tau \ddot{u}(\tau)\,d\tau$$

$$\dot{u}(\tau) = \dot{u}(t) + \underbrace{\ddot{u}(t).\tau + \frac{\Delta\ddot{u}}{\Delta t}.\frac{\tau^2}{2}}_{\Delta\dot{u}(\tau)}$$



$$u(\tau) = u(t) + \int_0^\tau \dot{u}(\tau)\,d\tau$$

$$u(\tau) = u(t) + \underbrace{\dot{u}(t).\tau + \ddot{u}(t).\frac{\tau^2}{2} + \frac{\Delta\ddot{u}}{\Delta t}.\frac{\tau^3}{6}}_{\Delta u(\tau)}$$

At $\tau = \Delta t$, the above equations for velocity and displacement becomes,

$$\dot{u}(t) = \ddot{u}(t)\Delta t + \frac{\Delta\ddot{u}}{\Delta t}\frac{\Delta t^2}{2}$$

$$u(t) = \dot{u}(t)\Delta t + \ddot{u}(t)\frac{\Delta t^2}{2} + \frac{\Delta\ddot{u}}{\Delta t}\frac{\Delta t^3}{6}$$

Re-writing the above two equations in terms of $\Delta u(t)$:

$$\Delta\ddot{u}(t) = \frac{6}{\Delta t^2}\Delta u(t) - \frac{6}{\Delta t}\dot{u}(t) - 3\ddot{u}(t)$$

$$\Delta\dot{u}(t) = \frac{3}{\Delta t}\Delta u(t) - 3\dot{u}(t) - \frac{\Delta t}{2}\ddot{u}(t)$$

The above two expressions are derived from the "linear acceleration "assumptions.

Let's assume that the calculation is made up to Time $= t$ and we are going to proceed to the next time stop, $t + \Delta t$.

Hence,

$u(t)$, $\dot{u}(t)$, $\ddot{u}(t)$ are known.

$\Delta u(t)$, $\Delta\dot{u}(t)$, $\Delta\ddot{u}(t)$ are to be determined.

3 unknowns in this time incremental steps,

In fact only $\Delta u(t)$ has to be determined since the remaining $\Delta\dot{u}(t)$ and $\Delta\ddot{u}(t)$ can be derived from $\Delta u(t)$ by the above two equations. In the other words, the "linear acceleration assumption" transforms the problem of 3 unknowns into the problem of one unknown $\Delta u(t)$.

Introducing the above two equations into the incremental form of governing equation of motion, we obtain,

$$m\left[\frac{6}{\Delta t^2}\Delta u(t) - \frac{6}{\Delta t}\dot{u}(t) - 3\ddot{u}(t)\right] + c(t)\left[\frac{3}{\Delta t}\Delta u(t) - 3\dot{u}(t) - \frac{\Delta t}{2}\ddot{u}(t)\right] + k(t)\Delta u(t) = \Delta p(t)$$

Re-writing the above equation, we get,

$$\tilde{k}(t)\,\Delta u(t) = \Delta\tilde{p}(t)$$

Where

$$\tilde{k}(t) = k(t) + \frac{6}{\Delta t^2}m + \frac{3}{\Delta t}c(t)$$

$$\Delta\tilde{p}(t) = \Delta p(t) + m\left[\frac{6}{\Delta t}\dot{u}(t) + 3\ddot{u}(t)\right] + c(t)\left[3\dot{u}(t) + \frac{\Delta t}{2}\ddot{u}(t)\right]$$

By this equation, $\Delta u(t)$ can be computed, and then the other two unknowns ($\Delta\dot{u}(t)$ and $\Delta\ddot{u}(t)$) can be derived from the expressions mentioned above.

Note:

Two assumptions are used in this step-by- step calculation.

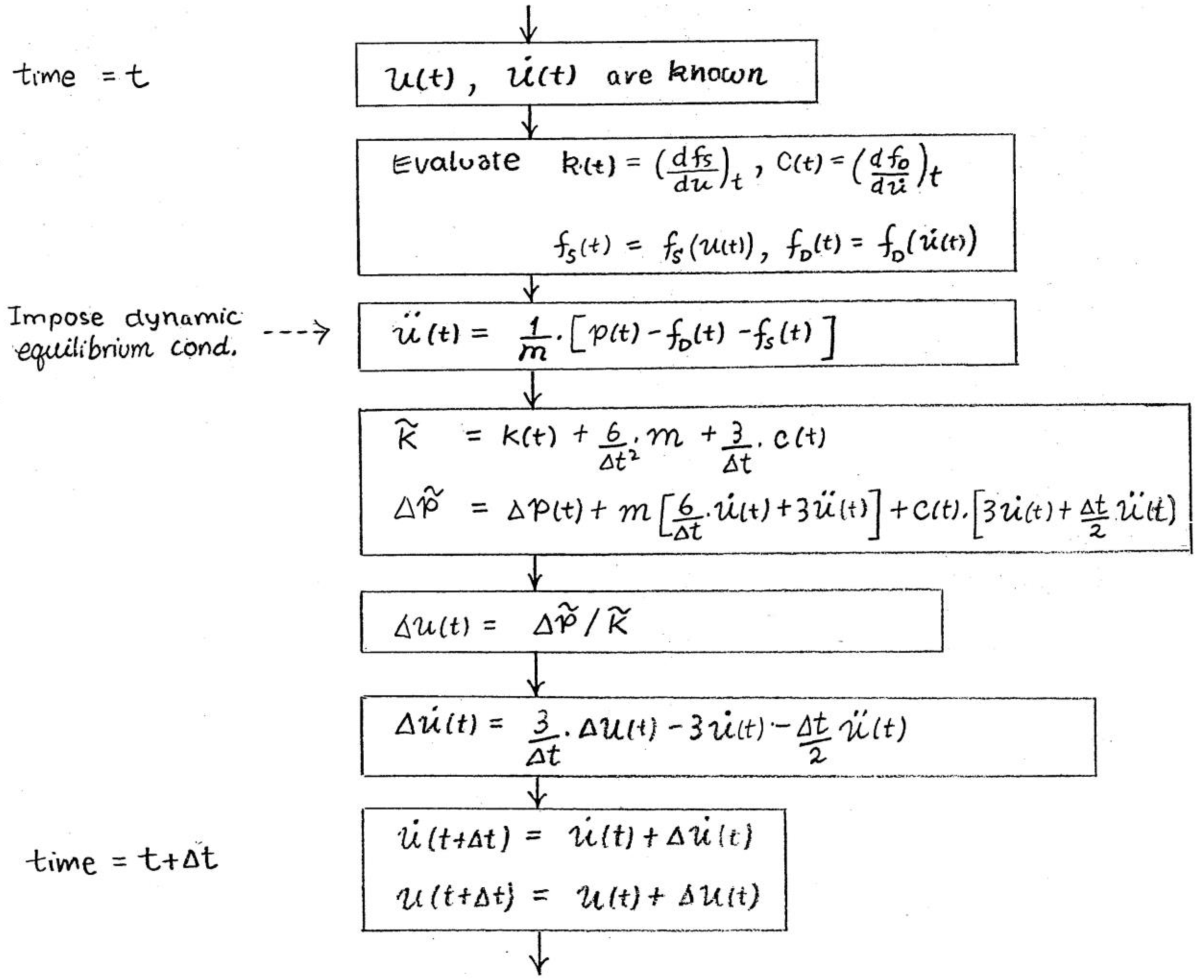
1) Within $\{t, t + \Delta t\}$, $\Delta f_D(t) = c(t)\Delta\dot{u}$ and $\Delta f_s(t) = k(t)\Delta u$
2) Within $\{t, t + \Delta t\}$, acceleration varies linearly

These assumptions are justified only when Δt is sufficiently small, small Δt → small error.

Although the error in each step is small, the error can be accumulate and become significant when the number of steps is large.

The accumulation should be avoided by imposing the dynamic equilibrium condition at each time step.

Calculation flow chart:



Additional notes:

1. Response of any SDF system with any prescribed nonlinear properties can be evaluated by “step-by-step integration”.
2. Response of linear SDOF system can also be evaluated by the step-by-step integration.
3. To determine Δt, we should consider :
    - The rate of variation of the applied loading $p(t)$
    - The nonlinearity of damping and stiffness properties.
    - The natural period of structure (T)

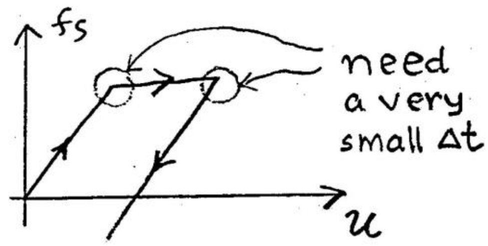
Rule of thumb:

$$\Delta t/T \leq 1/10$$

My suggestion:

$$\Delta t/T \leq 1/30$$

4. The step-by-step integration technique will be extended for the calculation of responses of nonlinear MDF system later. More attention will be paid on the accumulation of error–as it is a major factor in the determination of Δt.



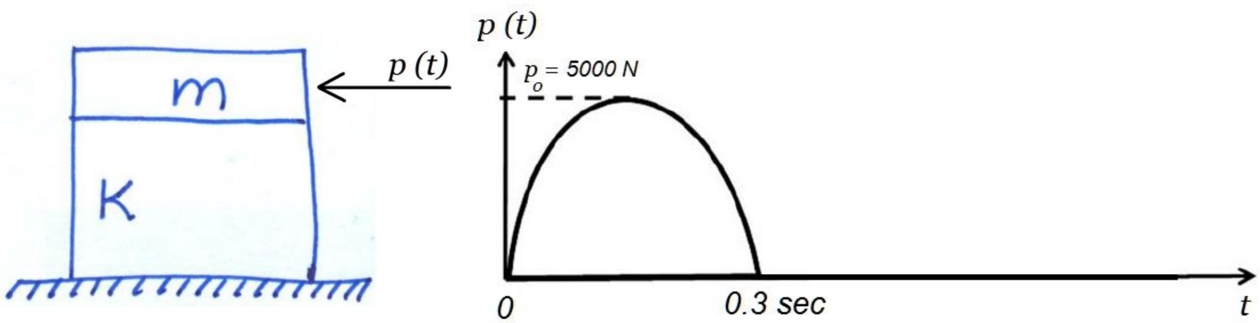
The choice of Δt also depends on the nonlinear properties of damping and stiffness

An Assignment Example: Response to General Dynamic Loading

Consider the following SDF system with $m = 2000\ Kg$, $k = 800{,}000\ N/m$ and $\xi = 0.027$ subjected to a dynamic loading $p(t)$ as shown below.

The loading function $p(t)$ is a half sine pulse as shown below.

$$p(t) = \begin{cases} p_o \sin(\pi t/t_d) & for\ t < t_d \\ 0 & for\ t \geq t_d \end{cases}$$



Consider at-rest initial conditions.

$t_d = 0.3\ sec$

$p_o = 5000\ N$

Find the displacement response for $0 \leq t \leq 2\ sec$ as well as the maximum response.

Note:

You can find the response using two approaches.

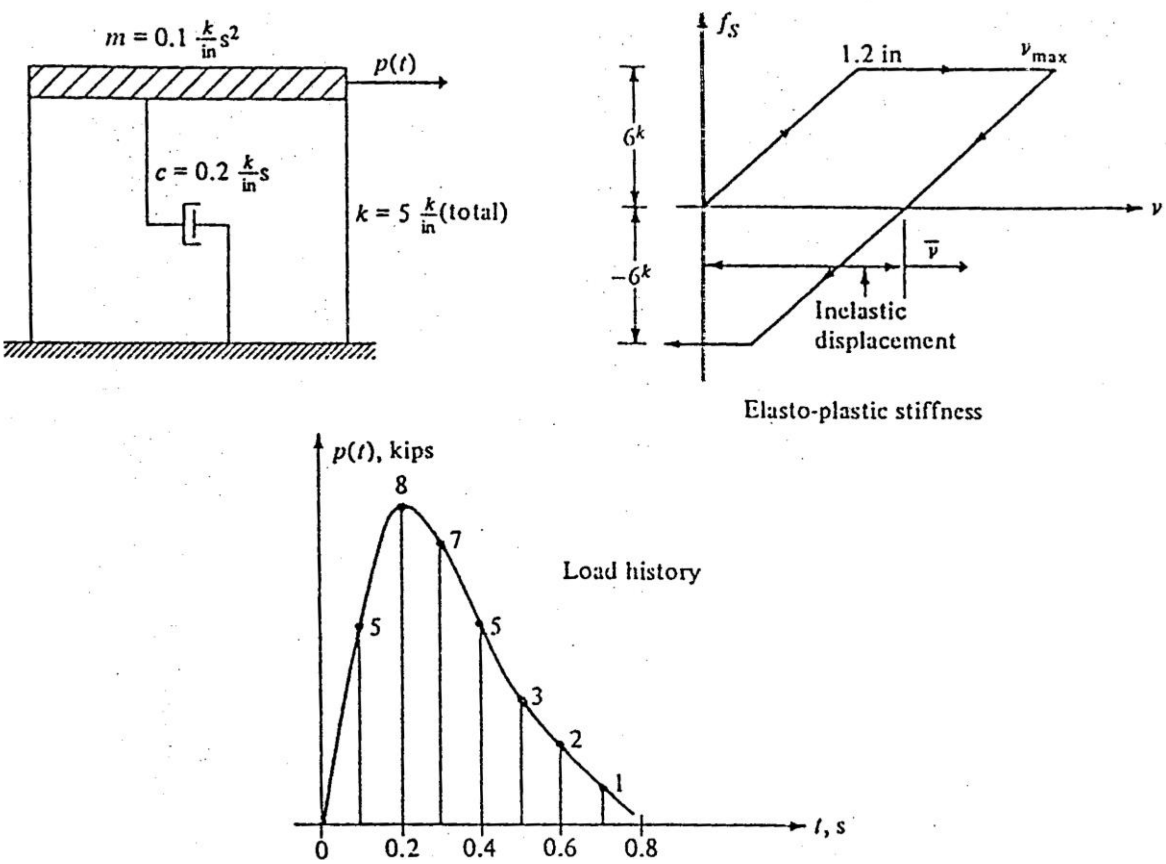
Option 1 (Analytical Approach):

Consider this loading as an impulse loading. Divide the response into two phases. Determine the particular solution for phase 1 (i.e. loading phase) and determine the homogeneous solution as free vibration response. Plot the response together in both phases. The maximum displacement response can be in any phase (phase 1 or phase 2).

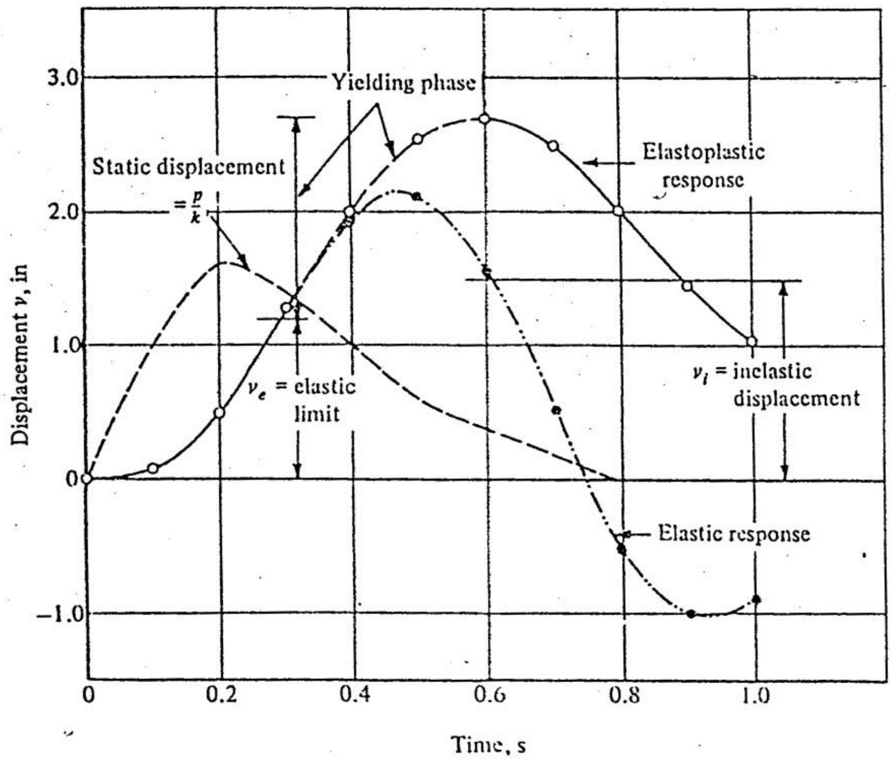
Option 2 (Numerical Approach):

In this approach, you have two choices i.e. either use "Duhamel's integral" (with numerical integration solution) or use the "Step-by-step Direct Integration" method. Determine the response from both procedures and compare.

Example:



An elastoplastic frame and dynamic loading



Comparison of elastoplastic with elastic response

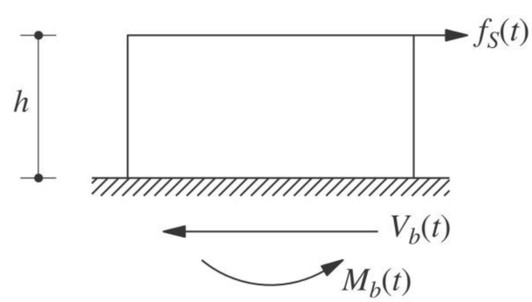
2.11. Earthquake Response of SDF Systems

The ground acceleration is defined by numerical values at discrete time instants. These time instants should be closely spaced to describe accurately the highly irregular variation of acceleration with time. Typically, the time interval is chosen to be 1/100 to 1/50 of a second, requiring 1500 to 3000 ordinates to describe the ground motion.

The governing equation of motion of an SDF system having a natural frequency ω and subjected to a ground motion $\ddot{u}_g(t)$ can be written as follows.

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = -\ddot{u}_g(t)$$

It is clear that for a given $\ddot{u}_g(t)$, the deformation response $u(t)$ of the system depends only on the natural frequency ω or natural period T of the system and its damping ratio ξ; writing formally, $u \equiv u(t, T, \xi)$. Thus any two systems having the same values of T and ξ will have the same deformation response $u(t)$ even though one system may be more massive than the other or one may be stiffer than the other.



Once the deformation response history $u(t)$ has been evaluated by dynamic analysis of the structure, the internal forces can be determined by static analysis of the structure at each time instant. Two methods to implement such analysis were mentioned in Chapter 1. Between them, the preferred approach in earthquake engineering is based on the concept of the equivalent static force f_s because it can be related to earthquake forces specified in building codes.

$$f_s(t) = ku(t)$$

Where k is the lateral stiffness of the frame. Expressing k in terms of the mass m gives

$$f_s(t) = m\omega^2 u(t) = mA(t)$$

Where $A(t) = \omega^2 u(t)$. Observe that the equivalent static force is m times $A(t)$, the pseudo-acceleration, not m times the total acceleration $\ddot{u}(t)$.

For the one-story frame the internal forces (e.g., the shears and moments in the columns and beam, or stress at any location) can be determined at a selected instant of time by static analysis of the structure subjected to the equivalent static lateral force $f_s(t)$ at the same time instant. Thus a static analysis of the structure would be necessary at each time instant when the responses are desired. In particular, the base shear $V_b(t)$ and the base overturning moment $M_b(t)$ are

$$V_b(t) = f_s(t)$$

$$M_b(t) = hf_s(t)$$

Where h is the height of the mass above the base. The above expressions can also be written as

$$V_b(t) = mA(t)$$

$$M_b(t) = hV_b(t)$$

2.11.1. The Concept of Response Spectrum

A plot of the peak value of a response quantity as a function of the natural vibration period T of the system, or a related parameter such as circular frequency ω or cyclic frequency f, is called the response spectrum for that quantity. Each such plot is for SDF systems having a fixed damping ratio ξ, and several such plots for different values of ξ are included to cover the range of damping values encountered in actual structures. Whether the peak response is plotted against f or T is a matter of personal preference.

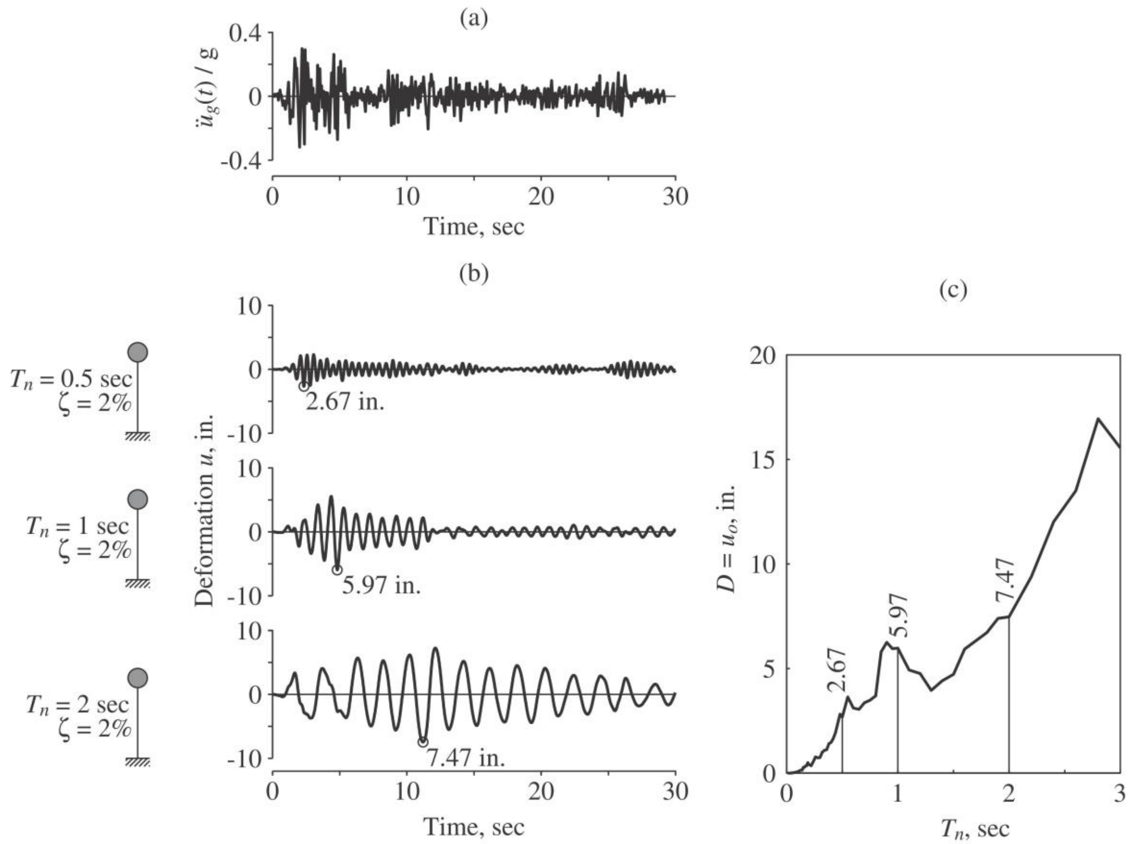
A variety of response spectra can be defined depending on the response quantity that is plotted. Consider the following peak responses:

$$u_o(T,\xi) = max|u(t,T,\xi)|$$

$$\dot{u}_o(T,\xi) = max|\dot{u}(t,T,\xi)|$$

$$\ddot{u}_o(T,\xi) = max|\ddot{u}(t,T,\xi)|$$

The deformation response spectrum is a plot of u_o against T for fixed ξ. A similar plot for $\dot{u}_o$ is the relative velocity response spectrum, and for $\ddot{u}_o$ is the acceleration response spectrum.



(a) Ground acceleration; (b) deformation response of three SDF systems with ξ = 2% and T = 0.5, 1, and 2 sec; (c) deformation response spectrum for ξ = 2%.

Pseudo-velocity response spectrum

Consider a quantity V for an SDF system with natural frequency ω related to its peak deformation $D \equiv u_o$ due to earthquake ground motion:

$$V = \omega D = \frac{2\pi}{T} D$$

The quantity V has units of velocity. It is related to the peak value of strain energy E_{so} stored in the system during the earthquake by the equation

$$E_{so} = \frac{mV^2}{2}$$

This relationship can be derived from the definition of strain energy and using the definition of V as follows:

$$E_{so} = \frac{ku_o^2}{2} = \frac{kD^2}{2} = \frac{k\left(\frac{V}{\omega}\right)^2}{2} = \frac{mV^2}{2}$$

The right side of above equation is the kinetic energy of the structural mass m with velocity V, called the peak pseudo-velocity. The prefix pseudo is used because V is not equal to the peak relative velocity $\dot{u}_o$, although it has the correct units.

The pseudo-velocity response spectrum is a plot of V as a function of the natural vibration period T, or natural vibration frequency f, of the system. For any ground motion, the peak pseudo-velocity V for a system with natural period T can be determined from above equation and the peak deformation D of the same system available from the response spectrum.

Pseudo-acceleration response spectrum

Consider a quantity A for an SDF system with natural frequency ω related to its peak deformation $D \equiv u_o$ due to earthquake ground motion:

$$A = \omega^2 D = \left(\frac{2\pi}{T}\right)^2 D$$

The quantity A has units of acceleration and is related to the peak value of base shear V_{bo} or the peak value of the equivalent static force f_{so}.

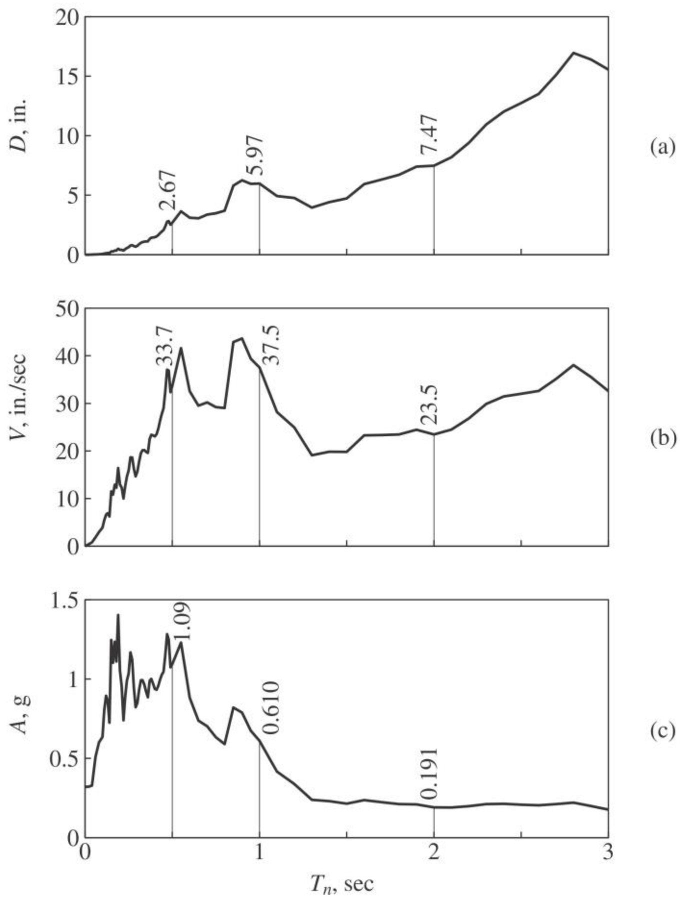
$$V_{bo} = f_{so} = mA$$

The peak base shear can be written in the form

$$V_{bo} = \frac{A}{g} w$$

where w is the weight of the structure and g the gravitational acceleration. When written in this form, A/g may be interpreted as the base shear coefficient or lateral force coefficient. It is used in building codes to represent the coefficient by which the structural weight is multiplied to obtain the base shear. Observe that the base shear is equal to the inertia force associated with the mass m undergoing acceleration A. This quantity A is generally different from the peak acceleration $\ddot{u}_o$ to of the system. It is for this reason that we call A the peak pseudo-acceleration; the prefix pseudo is used to avoid possible confusion with the true peak acceleration $\ddot{u}_o$.

The pseudo-acceleration response spectrum is a plot of A as a function of the natural vibration period T, or natural vibration frequency f, of the system. For any ground motion, the peak pseudo-acceleration A for a system with natural period T and damping ratio ξ can be determined from above equation and the peak deformation D of the system from the spectrum.



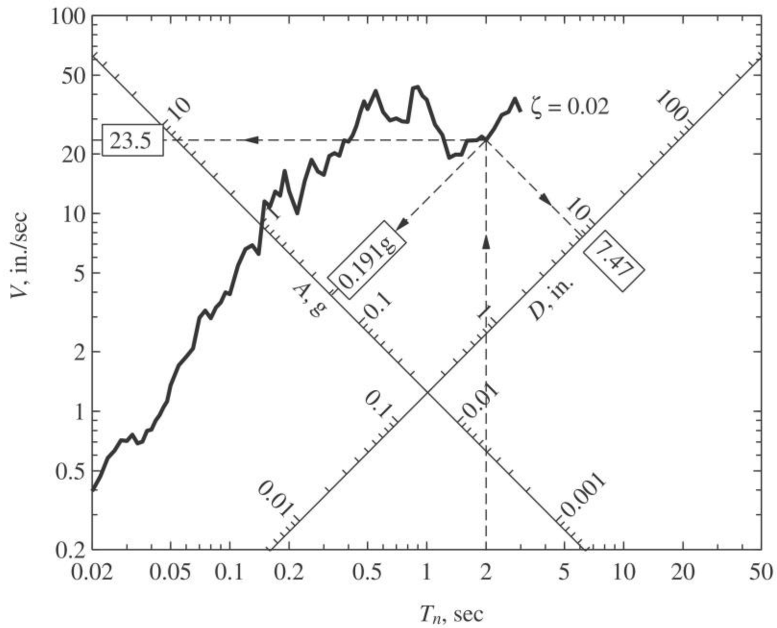
Response spectra ($\xi = 0.02$) for El Centro ground motion: (a) deformation response spectrum; (b) pseudo-velocity response spectrum; (c) pseudo-acceleration response spectrum.

Combined D-V-A response spectrum

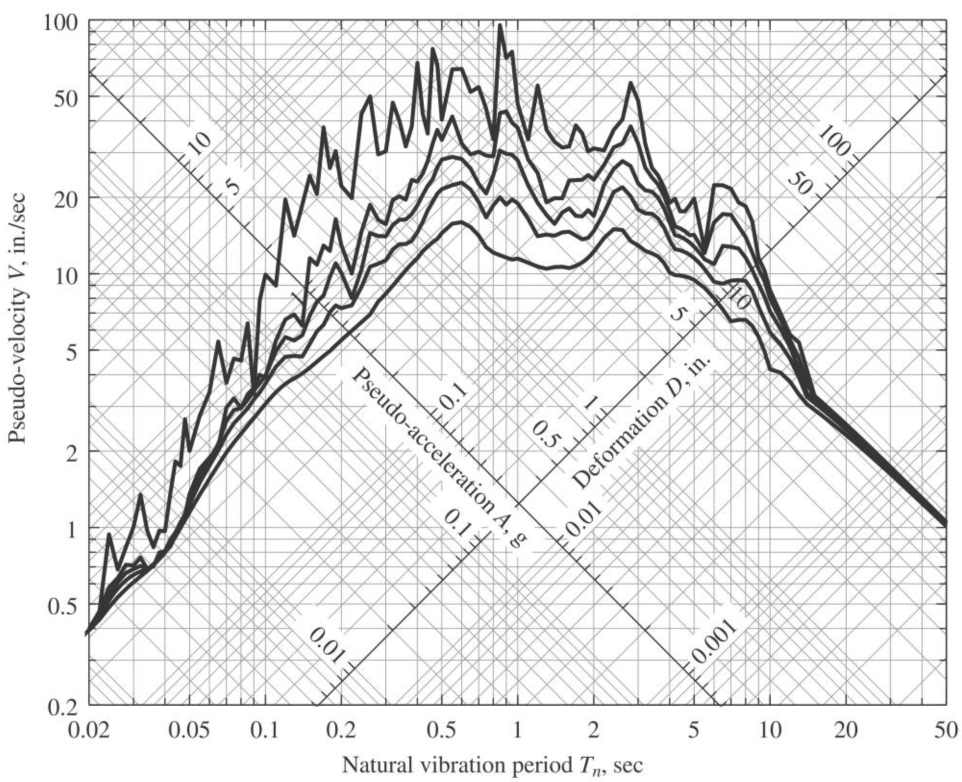
Each of the deformation, pseudo-velocity, and pseudo-acceleration response spectra for a given ground motion contains the same information, no more and no less. The three spectra are simply different ways of presenting the same information on structural response.

Knowing one of the spectra, the other two can be obtained by algebraic operations mentioned above.

Why do we need three spectra when each of them contains the same information? One of the reasons is that each spectrum directly provides a physically meaningful quantity. The deformation spectrum provides the peak deformation of a system. The pseudo-velocity spectrum is related directly to the peak strain energy stored in the system during the earthquake. The pseudo-acceleration spectrum is related directly to the peak value of the equivalent static force and base shear. The second reason lies in the fact that the shape of the spectrum can be approximated more readily for design purposes with the aid of all three spectral quantities rather than any one of them alone. For this purpose a combined plot showing all three of the spectral quantities is especially useful. This type of plot was developed for earthquake response spectra, apparently for the first time, by A. S. Veletsos and N. M. Newmark in 1960.



Combined D–V –A response spectrum for El Centro ground motion; ξ = 2%.



Combined D–V –A response spectrum for El Centro ground motion; ξ = 0, 2, 5, 10, and 20%.

Construction of Response Spectrum

The response spectrum for a given ground motion component $\ddot{u}_g(t)$ can be developed by implementation of the following steps:

1. Numerically define the ground acceleration $\ddot{u}_g(t)$; typically, the ground motion ordinates are defined every 0.02 sec.

2. Select the natural vibration period T and damping ratio ξ of an SDF system.

3. Compute the deformation response $u(t)$ of this SDF system due to the ground motion $\ddot{u}_g(t)$ by any of the numerical methods.

4. Determine u_o, the peak value of $u(t)$.

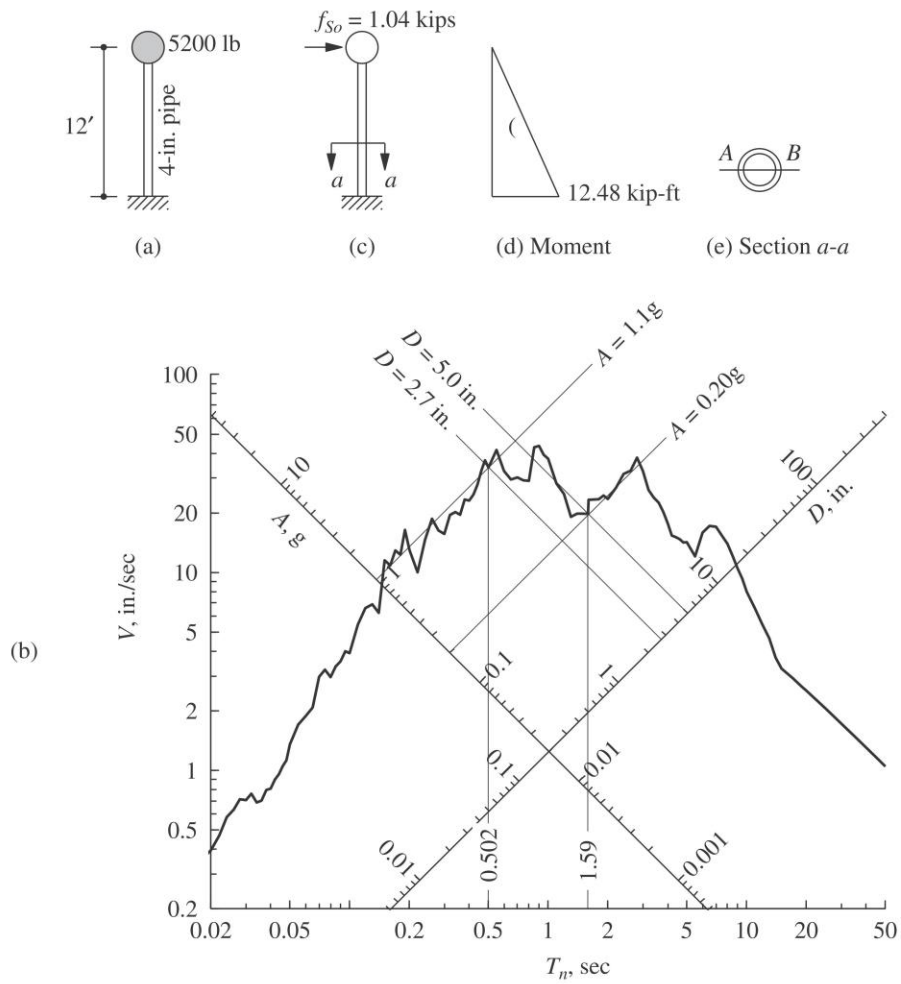
5. The spectral ordinates are $D = u_o$, $V = (2\pi/T)D$, and $A = (2\pi/T)^2 D$.

6. Repeat steps 2 to 5 for a range of T and ξ values covering all possible systems of engineering interest.

7. Present the results of steps 2 to 6 graphically to produce three separate spectra or a combined spectrum.

Solved Example: Application of Response Spectra to SDF Systems

A 12-ft-long vertical cantilever, a 4-in.-nominal-diameter standard steel pipe, supports a 5200-lb weight attached at the tip as shown in Figure below. The properties of the pipe are: outside diameter, $d_o = 4.5\ in.$, inside diameter $d_i = 4.026\ in.$, thickness $t = 0.237\ in.$, and second moment of cross-sectional area, $I = 7.23\ in^4$, elastic modulus $E = 29{,}000\ ksi$, and weight $= 10.79\ lb/foot\ length$. Determine the peak deformation and bending stress in the cantilever due to the El Centro ground motion. Assume that $\xi = 2\%$.



Taken from Chopra (2004)