CE 809 - Structural Dynamics

Lecture 6: Response of SDF Systems to General Dynamic Loading

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Response to General Dynamic Loading

Duhamel’s Integral (Convolution Integral)

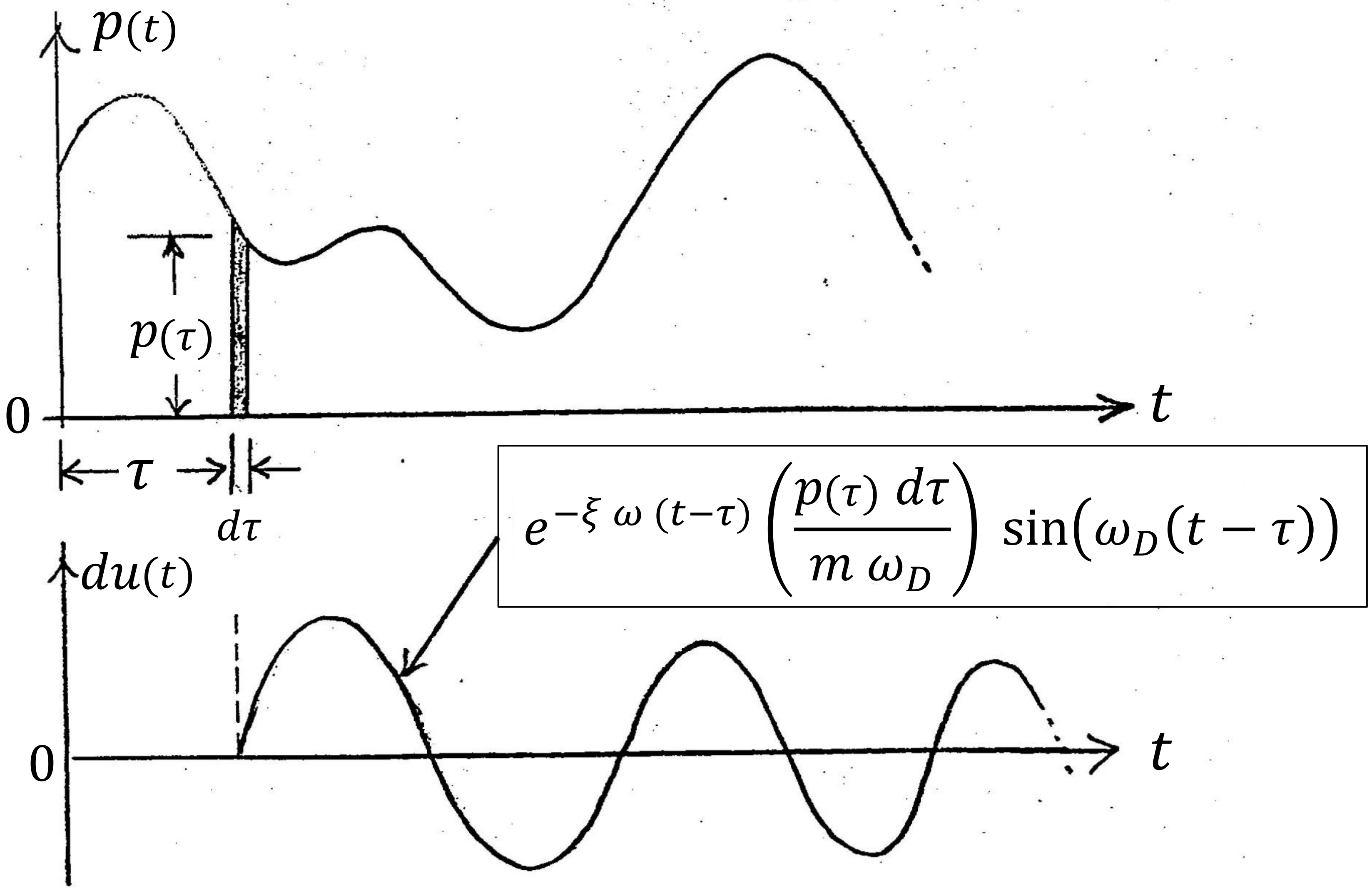
- Based on the **principle of superposition** (It considers the general dynamic loading as a series of short impulses).
- **Applicable only to linear systems**

Step-by-step Direct Integration Methods

- Based on the **direct numerical integration** of the governing equation of motion in incremental form.
- **Applicable to linear and nonlinear systems**

Duhamel’s Integral

Duhamel's Integral



$$e^{-\xi\,\omega\,(t-\tau)}\left(\frac{p(\tau)\,d\tau}{m\,\omega_D}\right)\,\sin\left(\omega_D(t-\tau)\right)$$

- A general dynamic loading = **A series of short Impulses**
- Each impulse produce its own (impulse) response
- **The sum of these impulse responses** = the response to the dynamic loading

Duhamel's Integral

Let $du(t;\tau)$ is the response of a linear dynamic system at time t due to impulse $p(\tau)\,d\tau$ at time τ.

$$du(t;\tau) \quad = \quad p(\tau)\,d\tau\,.\,h(t-\tau) \qquad \text{……… (1)}$$

Where

$$h\,(t-\tau) = \begin{cases} \dfrac{e^{-\xi\omega(t-\tau)}}{m\,\omega_D}\,\sin\big(\omega_D(t-\tau)\big), & t > \tau \\ 0\qquad\qquad\qquad\qquad\qquad\quad, & t \le \tau \end{cases} \qquad \text{……… (2)}$$

$h\,(t-\tau) =$ unit impulse response (or response to unit impulse applied at $t = \tau$).

$$du(t;\tau) = p(\tau)d\tau \,.\, h(t-\tau)$$

$$du(t;\tau = 0.\Delta t) = p(0.\Delta t)\, d\Delta t \,.\, h(t - 0.\Delta t)$$

$$du(t;\tau = 1.\Delta t) = p(1.\Delta t)\, d\Delta t \,.\, h(t - 1.\Delta t)$$

$$du(t;\tau = 2.\Delta t) = p(2.\Delta t)\, d\Delta t \,.\, h(t - 2.\Delta t)$$

$$du(t;\tau = i.\Delta t) = p(i.\Delta t)\, d\Delta t \,.\, h(t - i.\Delta t)$$

$$u(t) = \int_{\tau=0}^{\tau=t} p(\tau) \,.\, h(t-\tau)\, d\tau$$

Response to General Dynamic Loading - Duhamel's Integral

By means of **superposition** the total responsive $u(t)$ can be obtained by summing all impulse responses developed during the loading history.

$$u(t) = \int_{\tau=0}^{\tau=t} p(\tau) \, . \, h(t-\tau) \, d\tau \qquad \text{.......... (3)}$$

The integration is called "**Convolution Integral**" in general theory of mathematics and "**Duhamel's Integral**" in structural dynamics.

Response to General Dynamic Loading - Duhamel's Integral

In Equation (3), it is assumed that the structure is initially at-rest condition

That is $u(0) = 0, \;\; \dot{u}(0) = 0.$

For other cases, additional free vibration response must be added to the solution:

$$u(t) = e^{-\xi\,\omega\,t}\left[\frac{\dot{u}(0) + u(0)\,\xi\,\omega}{\omega_D}\sin(\omega_D t) + u(0)\cos(\omega_D t)\right] + \int_0^t p(\tau)\,h(t-\tau)\;d\tau \qquad \text{….. (4)}$$

In the following investigation, the initial at-rest condition is assumed.

Response to General Dynamic Loading - Duhamel's Integral

$$u(t) = \frac{1}{m\,\omega_D}\int_0^t p(\tau)\,.\,\underbrace{e^{-\xi\,\omega(t-\tau)}}_{e^{-\xi\,\omega\,t}\,.\,e^{\xi\,\omega\,\tau}}\,\underbrace{\sin\big(\omega_D(t-\tau)\big)}_{\sin(\omega_D t)\cos(\omega_D \tau) - \cos(\omega_D t)\sin(\omega_D \tau)}\,d\tau$$

Therefore,

$$u(t) = \frac{1}{m\,\omega_D}\int_0^t p(\tau).\,e^{-\xi\omega t}e^{\xi\omega\tau}\left[\sin(\omega_D t)\cos\big(\omega_D\tau\big) - \cos(\omega_D t)\sin(\omega_D\tau)\right]d\tau$$

Response to General Dynamic Loading - Duhamel's Integral

By rearranging the terms, we get

$$u\ (t)= \left[\frac{e^{-\xi\omega t}}{m\ \omega_D} \int_0^t p\ (\tau).\ e^{\xi\omega\tau} \cos(\omega_D \tau)\ d\tau \right] \sin(\omega_D t) - \left[\frac{e^{-\xi\omega t}}{m\ \omega_D} \int_0^t p(\tau)\ .\ e^{\xi\omega\tau} \sin(\omega_D \tau)\ d\tau \right] \cos(\omega_D t)$$

So we can write

$$u(t) \ = A(t) \sin \omega_D t - B(t) \cos \omega_D t$$

Where

$$A(t) \ = \frac{e^{-\xi\omega t}}{m\ \omega_D} \left(\int_0^t p(\tau)\ .\ e^{\xi\omega\tau}.\ \cos(\omega_D \tau)\ d\tau \right)$$

$$B(t) \ = \frac{e^{-\xi\omega t}}{m\ \omega_D} \left(\int_0^t p(\tau)\ .\ e^{\xi\omega\tau}.\sin(\omega_D \tau)\ d\tau \right)$$

Response to General Dynamic Loading - Duhamel's Integral

For undamped case,

$$A(t) = \frac{1}{m\,\omega}\left(\int_0^t p(\tau)\,.\,\cos(\omega\,\tau)\,d\tau\right)$$

$$B(t) = \frac{1}{m\,\omega}\left(\int_0^t p(\tau)\,.\,\sin(\omega\,\tau)\,d\tau\right)$$

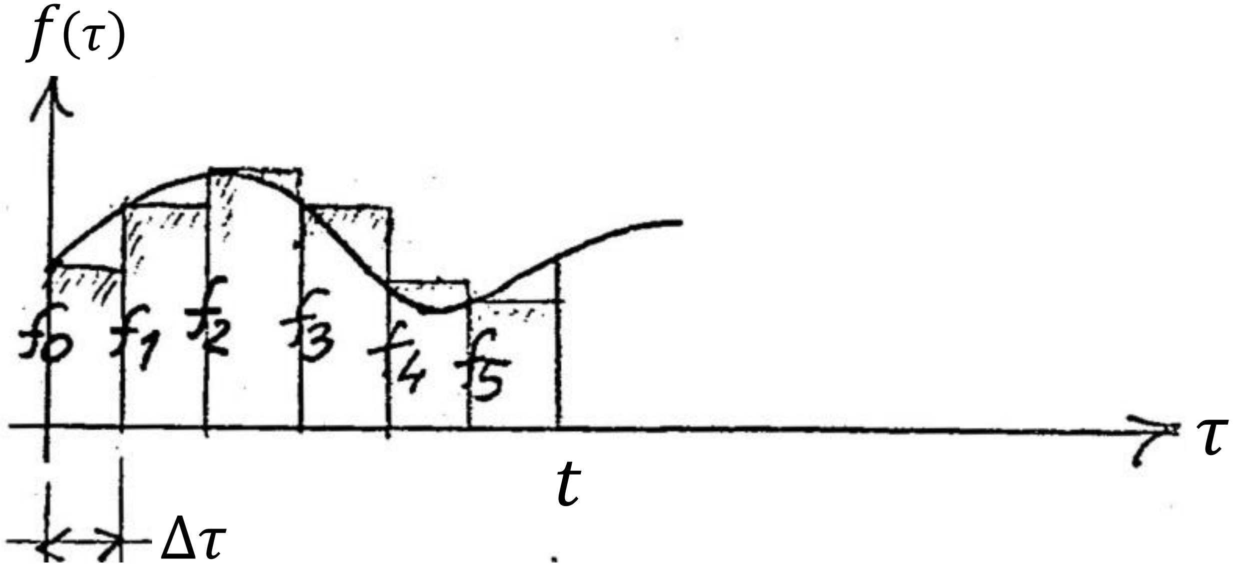
The terms in parenthesis for (both damped and undamped cases) need **"numerical integration"**.

Numerical Integration

Simple Summation:

$$\int_0^t f(\tau)\, d\tau \;\cong\; \Delta\tau\,(f_0 + f_1 + f_2 + f_3 + \cdots f_{N-1})$$

Where $f_i = f(\tau = i.\Delta\tau)$, and $\Delta\tau = t/N$

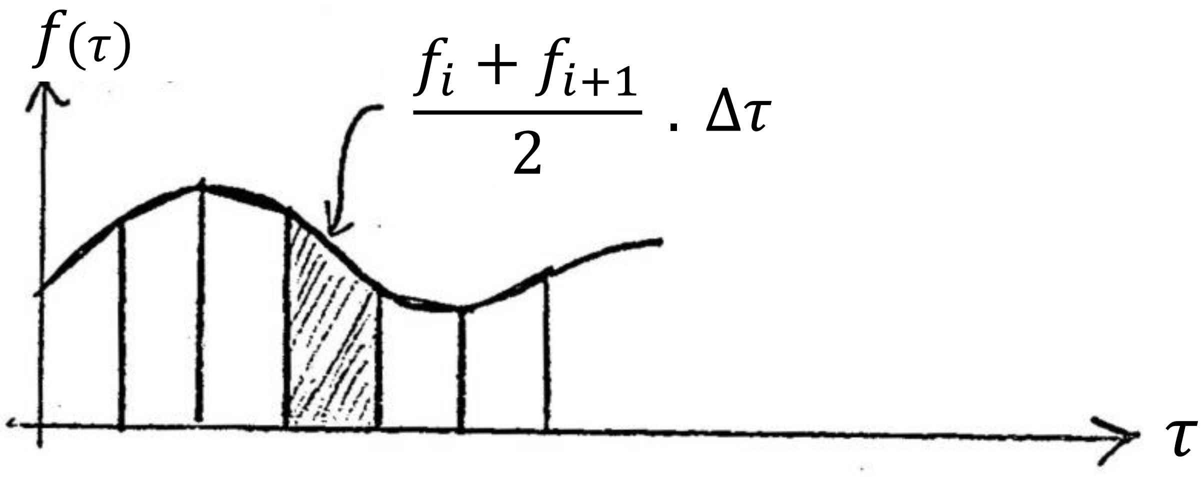


Numerical Integration

Trapezoidal Rule:

$$\int_0^t f(\tau)\, d\,\tau \quad \cong \quad \frac{\Delta\tau}{2}\left(f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots 2f_{N-1} + f_N\right)$$

Where $f_i = f(i.\Delta\tau)$, and $\Delta\tau = t/N$



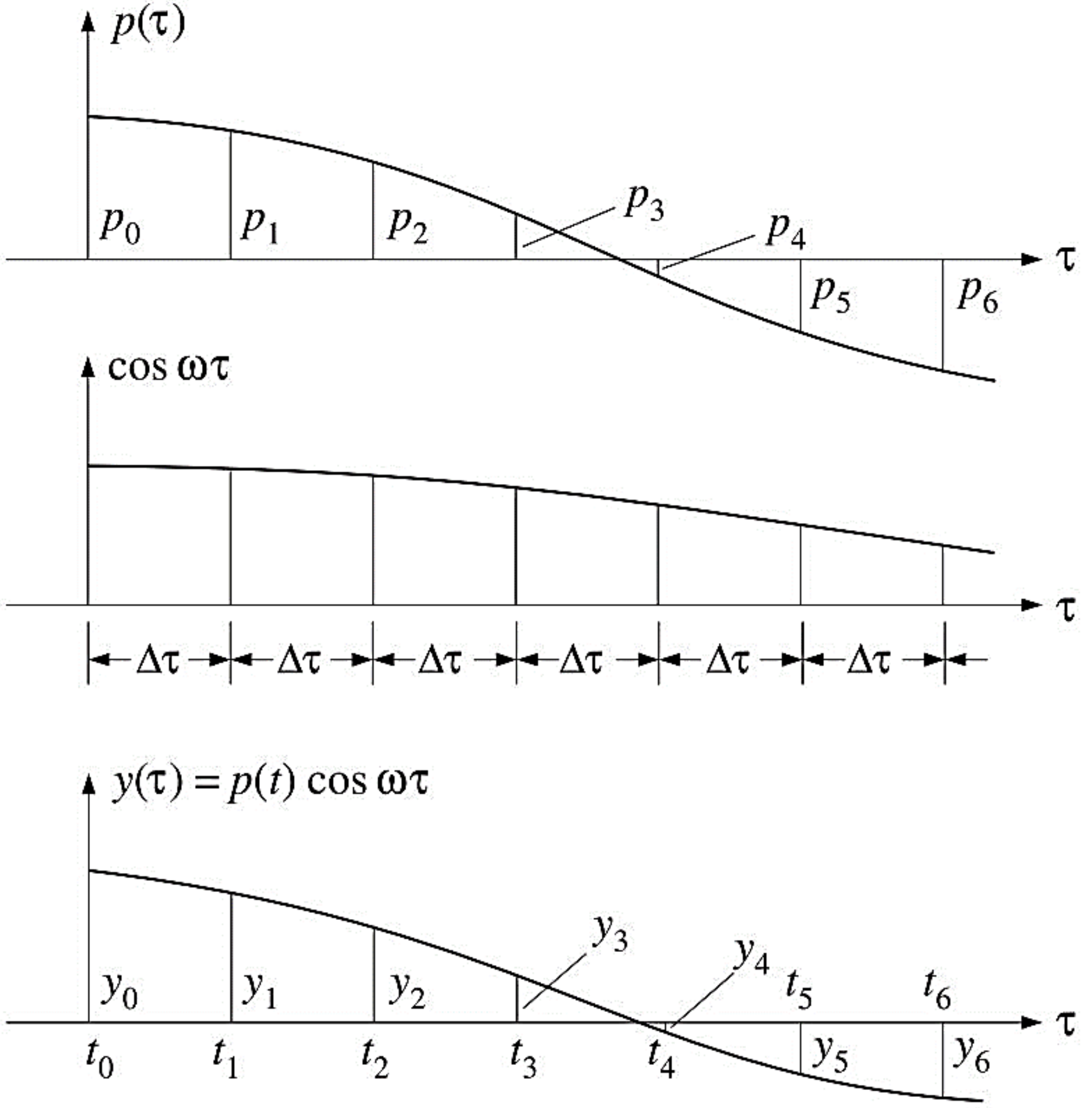
Solving Duhamel's Integral using Numerical Integration

- For example, consider the numerical integration of a function $y(\tau) = p(\tau)\cos\omega\tau$ as required to find $A(t)$ in Duhamel's Integral.

- For convenience of numerical calculation, the function $y(\tau)$ is evaluated at equal time increments $\Delta\tau$ as shown in Figure.

- The integral A_N can now be obtained approximately by summing the ordinates, after multiplying by weighting actors that depend on the numerical integration scheme being used.

Source: Clough and Penzien (2003)



Solving Duhamel's Integral using Numerical Integration

Undamped Systems

Simple summation:

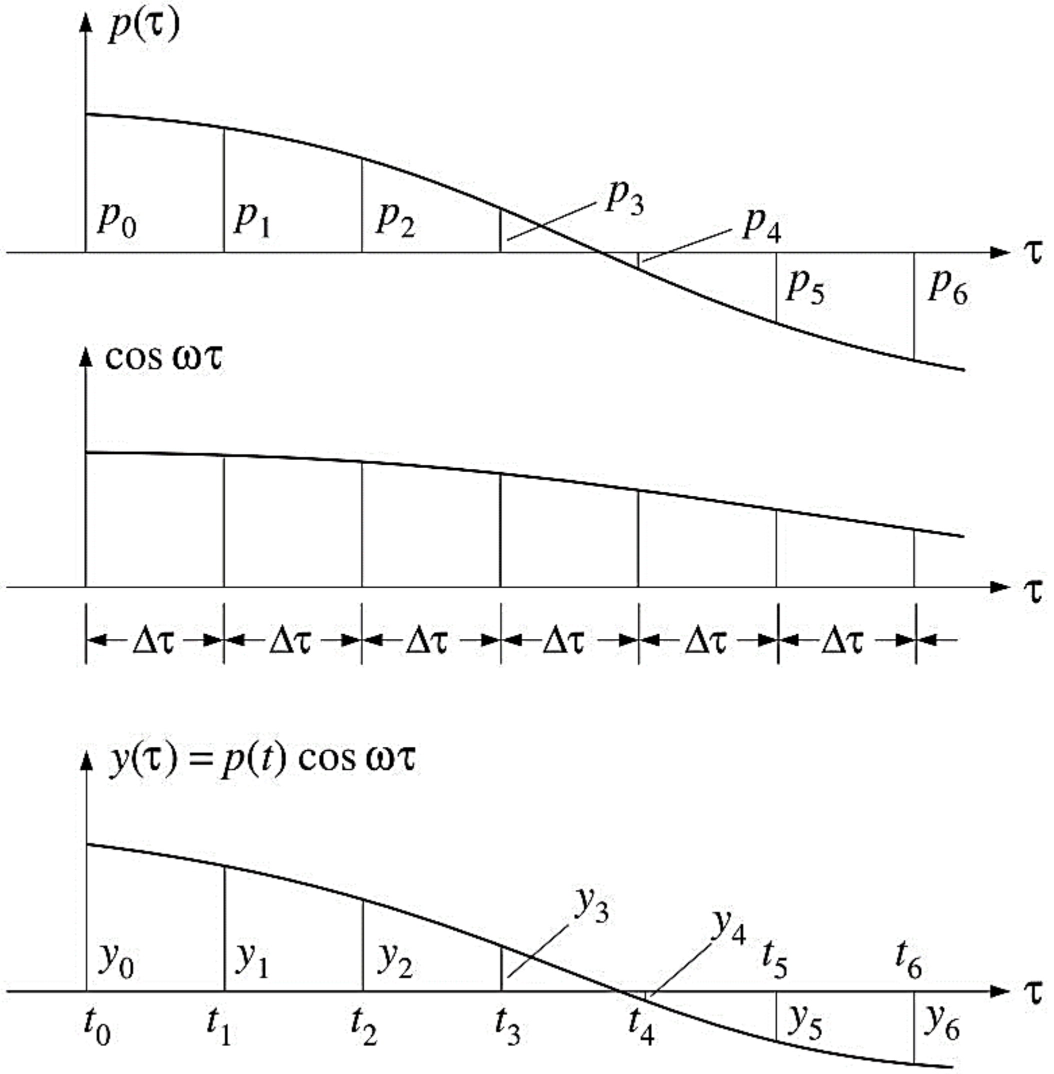
$$A_N = \frac{\Delta\tau}{m\,\omega}\left[y_0 + y_1 + y_2 + \cdots + y_{N-1}\right]$$

Trapezoidal rule:

$$A_N = \frac{\Delta\tau}{2\,m\,\omega}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{N-1} + y_N\right]$$

Simpson's rule:

$$A_N = \frac{\Delta\tau}{3\,m\,\omega}\left[y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-1} + y_N\right]$$



Source: Clough and Penzien (2003)

It is more efficient to write these equations in their recursive forms:

Simple summation:

$$A_N = A_{N-1} + \frac{\Delta\tau}{m\,\omega}[y_{N-1}] \qquad N = 1, 2, 3, \ldots$$

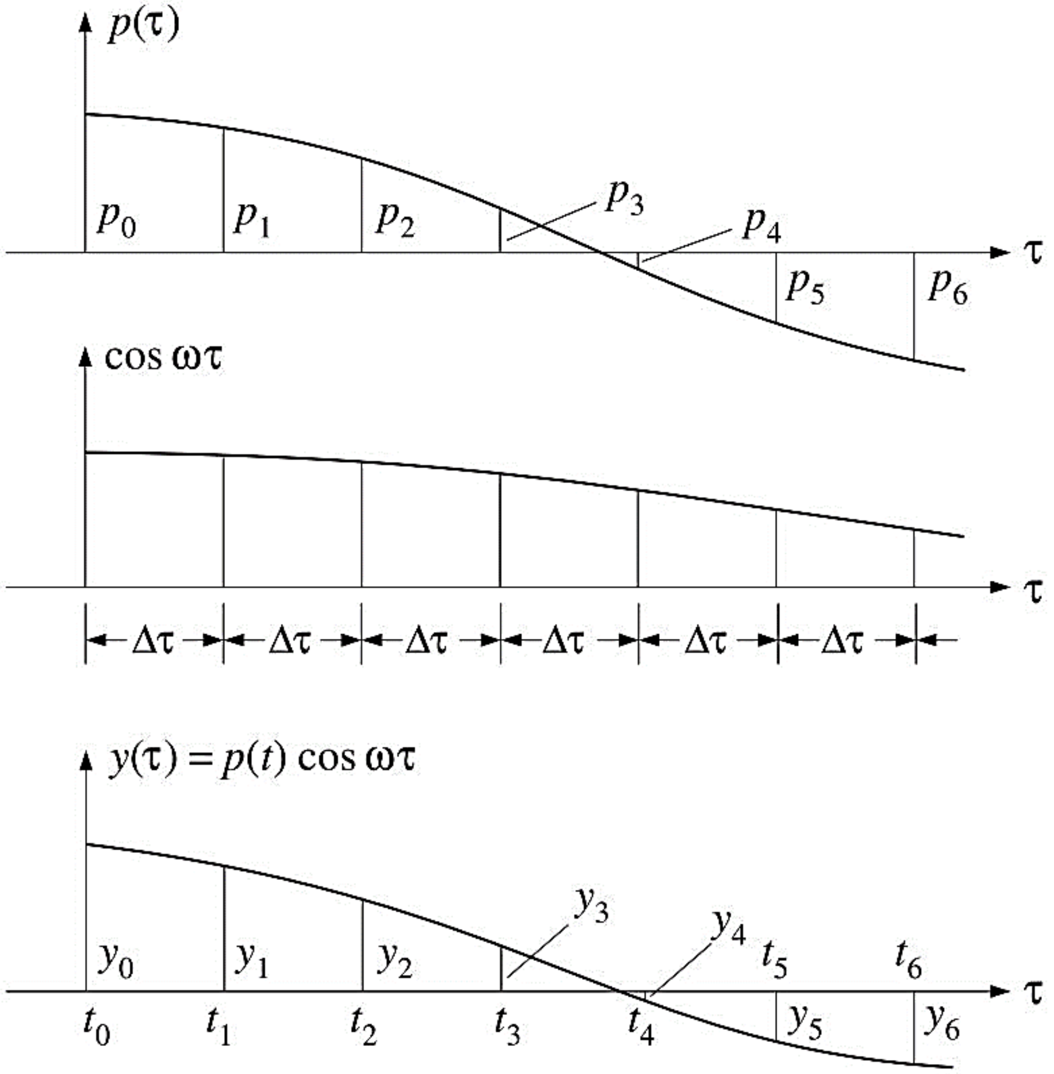
Trapezoidal rule:

$$A_N = A_{N-1} + \frac{\Delta\tau}{2\,m\,\omega}[y_{N-1} + y_N] \qquad N = 1, 2, 3, \ldots$$

Simpson's rule:

$$A_N = A_{N-1} + \frac{\Delta\tau}{3\,m\,\omega}[y_{N-2} + 4y_{N-1} + y_N] \qquad N = 2, 4, 6, \ldots$$

Such that $A_0 = 0$



Source: Clough and Penzien (2003)

Solving Duhamel's Integral using Numerical Integration

Damped Systems

Simple summation:

$$A_N = e^{-\xi\,\omega\,\Delta\tau}\,A_{N-1} + \frac{\Delta\tau}{m\,\omega_D}\left[y_{N-1}\right] e^{-\xi\,\omega\,\Delta\tau} \qquad N = 1, 2, 3, \ldots$$

Trapezoidal rule:

$$A_N = e^{-\xi\,\omega\,\Delta\tau}\,A_{N-1} + \frac{\Delta\tau}{2\,m\,\omega_D}\left[y_{N-1}\,e^{-\xi\,\omega\,\Delta\tau} + y_N\right] \qquad N = 1, 2, 3, \ldots$$

Simpson's rule:

$$A_N = e^{-2\,\xi\,\omega\,\Delta\tau}\,A_{N-1} + \frac{\Delta\tau}{3\,m\,\omega_D}\left[y_{N-2}\,e^{-2\,\xi\,\omega\,\Delta\tau} + 4\,y_{N-1}\,e^{-\xi\,\omega\,\Delta\tau} + y_N\right] \qquad N = 2, 4, 6, \ldots$$

Solving Duhamel's Integral using Numerical Integration

- The evaluation of $B(t)$ can be carried out in the same manner, however, in doing so, the definition of $y(\tau)$ must be changed to $y(\tau) = p(\tau)\sin(\omega\tau)$.

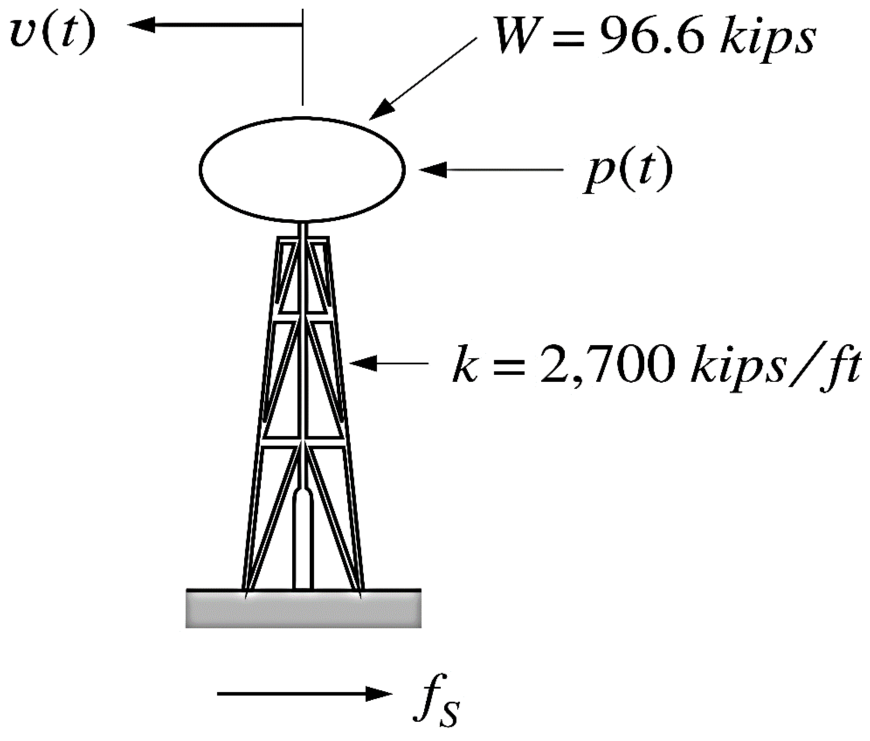
- Having calculated the values of A_N and B_N for successive values of N, the corresponding values of response u_N are obtained using

$$u_N = A_N \sin(\omega t_N) - B_N \cos(\omega t_N)$$

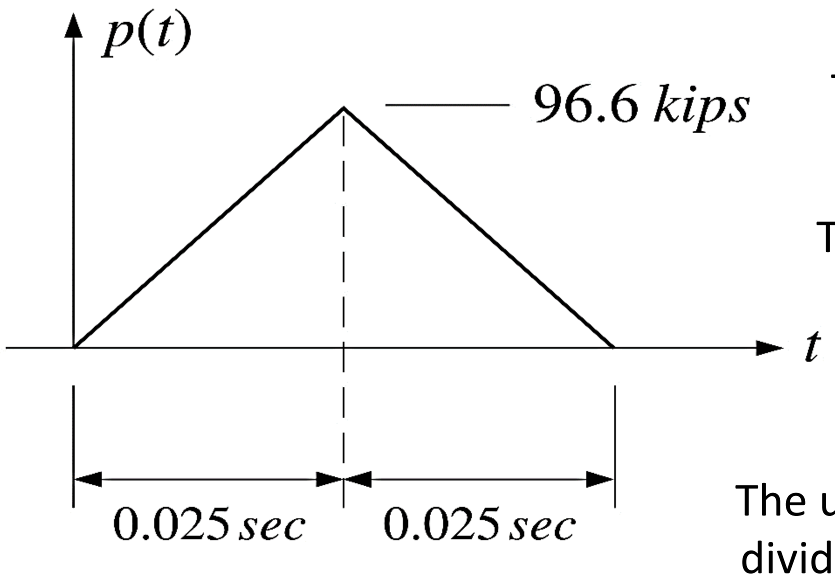
Source: Clough and Penzien (2003)

Numerical Example

Taken from Clough and Penzien (2003)



A water tower subjected to blast load



Loading history

The unit of force is kips.

1.0 kip = 1000 lbs

The unit of displacement is ft.

The unit of velocity is ft/s.

The unit of acceleration is ft/s^2.

1.0 g = 32.2 ft/s^2

The unit of mass is the unit of force divided by the unit of acceleration.

Kip- s^2 /ft

The unit of stiffness is the unit of force divided by the unit of displacement.

Kip/ft

Numerical Example

Taken from Clough and Penzien (2003)

TABLE E6-1

Numerical Duhamel integral analysis without damping

N	t_N sec	p_N kips (1)	$\sin 30t_N$ (2)	$\cos 30t_N$ (3)	y_N (1)×(3) kips (4)	y_{N-1} (5)	y_{N-2} (6)	$M_1 \times$ (5) (7)	$M_2 \times$ [(6)+(9)] (8)	$\frac{\bar{A}_{N-2}}{F}$ (9)	$\frac{\bar{A}_N}{F}$ (4)+(7)+(8) (10)	y_N (1)×(2) kips (11)	y_{N-1} (12)	y_{N-2} (13)	$M_1 \times$ (12) (14)	$M_2 \times$ [(13)+(16)] (15)	$\frac{\bar{B}_{N-2}}{F}$ (16)	$\frac{\bar{B}_N}{F}$ (11)+(14)+(15) (17)	(10)×(2) (18)	(17)×(3) (19)	(18)−(19) (20)	v_N F×(20) ft (21)	fs_N k×(21) kips (22)
0	0.000	0	0	1.000	0	—	—	—	—	—	0	0	—	—	—	—	—	0	0	0	0	0	0
1	0.005	19.32	0.149	0.989	19.1	0	—	—	—	—	—	2.88	0	—	—	—	—	—	—	—	—	—	—
2	0.010	38.64	0.296	0.955	36.9	19.1	0	76.4	0	0	113.3	11.4	2.88	0	11.5	0	0	22.9	33.5	21.9	11.6	0.0002	0.54
3	0.015	57.96	0.435	0.900	52.2	36.9	19.1	—	—	—	—	25.2	11.4	2.88	—	—	—	—	—	—	—	—	—
4	0.020	77.28	0.565	0.825	63.8	52.2	36.9	208.8	150.2	113.3	422.8	43.7	25.2	11.4	100.8	34.3	22.9	178.8	239	148	91	0.0017	4.60
5	0.025	96.60	0.682	0.732	70.7	63.8	52.2	—	—	—	—	65.9	43.7	25.2	—	—	—	—	—	—	—	—	—
6	0.030	77.28	0.783	0.622	48.1	70.7	63.8	282.8	486.6	422.8	817.5	60.5	65.9	43.7	263.6	222.5	178.8	546.6	640	340	300	0.0056	15.1
7	0.035	57.96	0.867	0.498	28.9	48.1	70.7	—	—	—	—	50.3	60.5	65.9	—	—	—	—	—	—	—	—	—
8	0.040	38.64	0.932	0.362	14.0	28.9	48.1	115.6	865.6	817.5	995.2	36.0	50.3	60.5	201.2	607.1	546.6	844.3	928	306	622	0.015	31.0
9	0.045	19.32	0.976	0.219	4.23	14.0	28.9	—	—	—	—	18.9	36.0	50.3	—	—	—	—	—	—	—	—	—
10	0.050	0	0.997	0.0707	0	4.23	14.0	16.9	1009	995.2	1026	0	18.9	36.0	75.6	880.3	844.3	955.9	1023	67.6	955	0.0177	47.8

$\omega = \sqrt{\frac{kg}{W}} = 30\ rad/sec \qquad \Delta\tau = 0.0005\ sec \qquad M_1 = 4 \qquad M_2 = 1 \qquad F \equiv \frac{\Delta\tau}{3m\omega} = 1.852 \times 10^{-5}\ ft/kip \qquad k = 2700\ kips/ft$

TABLE E6-2

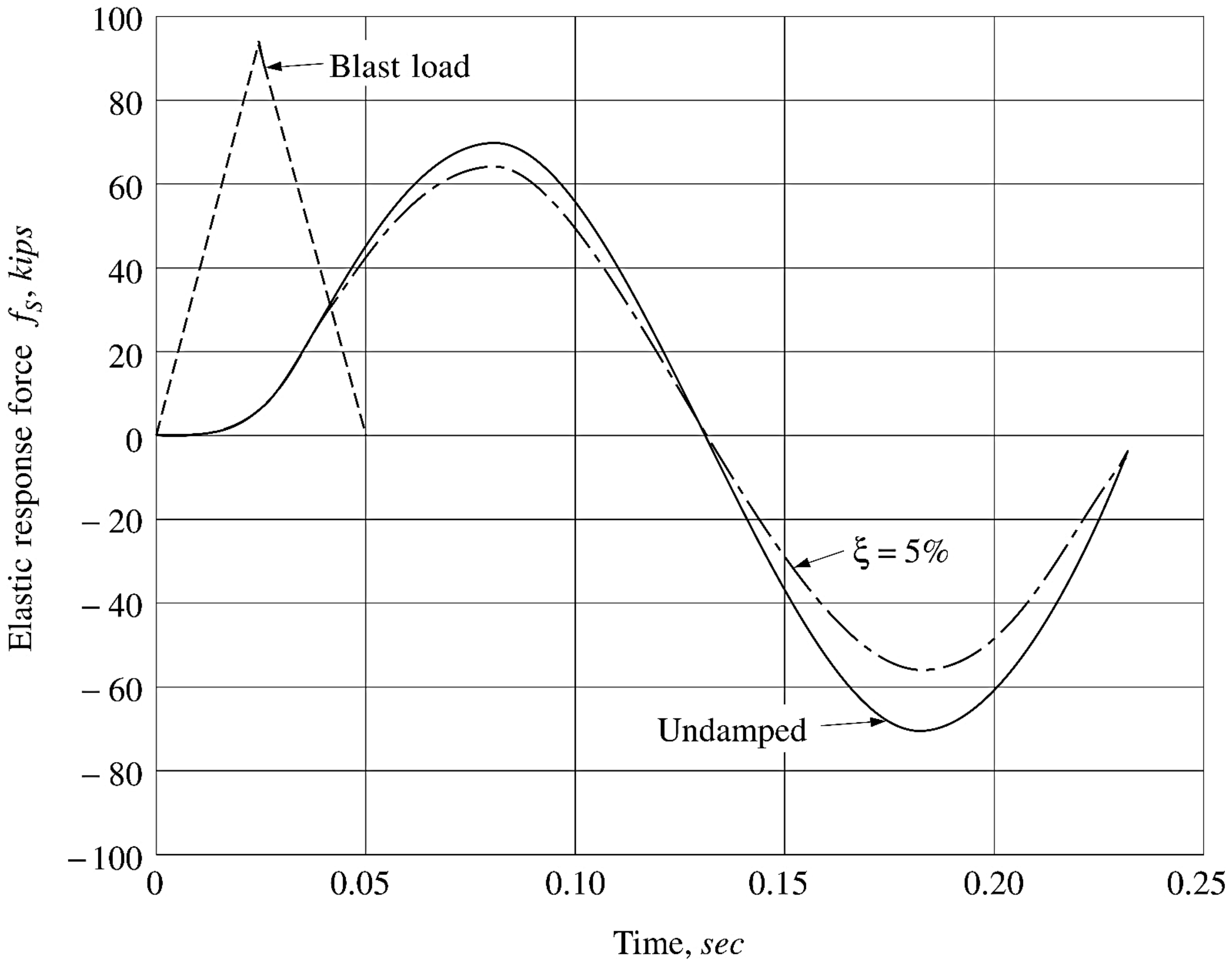
Numerical Duhamel integral analysis including damping

N	t_N sec	p_N kips (1)	$\sin 30t_N$ (2)	$\cos 30t_N$ (3)	y_N (1)×(3) kips (4)	y_{N-1} (5)	y_{N-2} (6)	$M_1 \times$ (5) (7)	$M_2 \times$ [(6)+(9)] (8)	$\frac{A_{N-2}}{F}$ (9)	$\frac{A_N}{F}$ (4)+(7)+(8) (10)	y_N (1)×(2) kips (11)	y_{N-1} (12)	y_{N-2} (13)	$M_1 \times$ (12) (14)	$M_2 \times$ [(13)+(16)] (15)	$\frac{B_{N-2}}{F}$ (16)	$\frac{B_N}{F}$ (11)+(14)+(15) (17)	(10)×(2) (18)	(17)×(3) (19)	(18)−(19) (20)	v_N F×(20) ft (21)	f_{S_N} k×(21) kips (22)
0	0.000	0	0	1.000	0	—	—	—	—	—	0	0	—	—	—	—	—	0	0	0	0	0	0
1	0.005	19.32	0.149	0.989	19.1	0	—	—	—	—	—	2.88	0	—	—	—	—	—	—	—	—	—	—
2	0.010	38.64	0.296	0.955	36.9	19.1	0	75.8	0	0	112.7	11.4	2.88	0	11.4	0	0	22.8	33.3	21.8	11.5	0.0002	0.58
3	0.015	57.96	0.435	0.900	52.2	36.9	19.1	—	—	—	—	25.2	11.4	2.88	—	—	—	—	—	—	—	—	—
4	0.020	77.28	0.565	0.825	63.8	52.2	36.9	207.2	147.4	112.7	418.4	43.7	25.2	11.4	100.0	33.7	22.8	177.4	236	146	90	0.0017	4.50
5	0.025	96.60	0.682	0.732	70.7	63.8	52.2	—	—	—	—	65.9	43.7	25.2	—	—	—	—	—	—	—	—	—
6	0.030	77.28	0.783	0.622	48.1	70.7	63.8	280.7	475.0	418.4	803.8	60.5	65.9	43.7	261.6	217.8	177.4	539.9	629	336	293	0.0054	14.65
7	0.035	57.96	0.867	0.498	28.9	48.1	70.7	—	—	—	—	50.3	60.5	65.9	—	—	—	—	—	—	—	—	—
8	0.040	38.64	0.932	0.362	14.0	28.9	48.1	114.7	839.1	803.8	967.8	36.0	50.3	60.5	199.7	591.4	539.9	827.1	902	299	603	0.0112	30.2
9	0.045	19.32	0.976	0.219	4.23	14.0	28.9	—	—	—	—	18.9	36.0	50.3	—	—	—	—	—	—	—	—	—
10	0.050	0	0.997	0.0707	0	4.23	14.0	16.8	967.1	967.8	983.9	0	18.9	36.0	75.0	850.1	827.1	925.1	981	65.4	915	0.0169	45.8
11	0.055	0	0.997	−0.0791	0	0	4.23	—	—	—	—	0	0	18.9	—	—	—	—	—	—	—	—	—
12	0.060	0	0.974	−0.227	0	0	0	0	969.1	983.9	969.1	0	0	0	0	911.2	925.1	911.2	900	−206	1106	0.0205	55.4
13	0.065	0	0.929	−0.370	0	0	0	—	—	—	—	0	0	0	—	—	—	—	—	—	—	—	—
14	0.070	0	0.863	−0.505	0	0	0	0	954.6	969.1	0	0	0	0	0	897.5	911.2	897.5	824	−453	1277	0.0236	63.9
15	0.075	0	0.778	−0.628	0	0	0	—	—	—	—	0	0	0	—	—	—	—	—	—	—	—	—
16	0.080	0	0.675	−0.737	0	0	0	0	940.3	954.6	940.3	0	0	0	0	884.0	897.5	884.0	635	−651.5	1286	0.0238	64.3
17	0.085	0	0.558	−0.830	0	0	0	—	—	—	—	0	0	0	—	—	—	—	—	—	—	—	—
18	0.090	0	0.427	−0.904	0	0	0	0	926.2	940.3	926.2	0	0	0	0	870.7	884.0	870.7	395	−787	1182	0.0219	59.1

$\omega = \sqrt{\frac{kg}{W}} = 30\ rad/sec$ $\quad \Delta\tau = 0.005\ sec$ $\quad M_1 = 4\exp(-\xi\omega\Delta\tau) = 3.97$ $\quad M_2 = \exp(-2\xi\omega\Delta\tau) = 0.985$ $\quad F = \frac{\Delta\tau}{3m\omega} = 1.852 \times 10^{-5}\ ft/kip$ $\quad k = 2700\ kips/ft$

Numerical Example

Taken from Clough and Penzien (2003)



Duhamel's Integral

- The concept of convolution integral will be used again later when we study the response of structures to **random loadings** from statistical view point (random vibration theory).

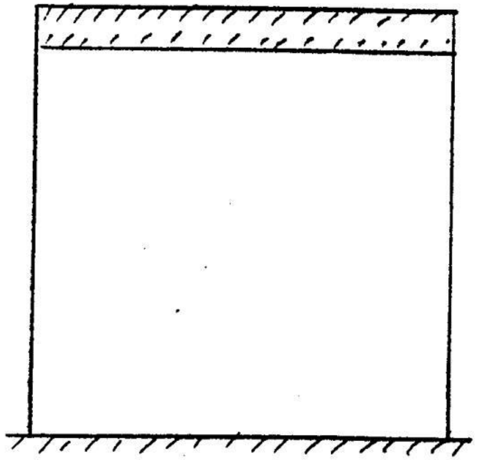
- The Convolution Integral is derived based on the principle of superposition. So, it is applicable only for the **response analysis of "linear systems"**.

Applications

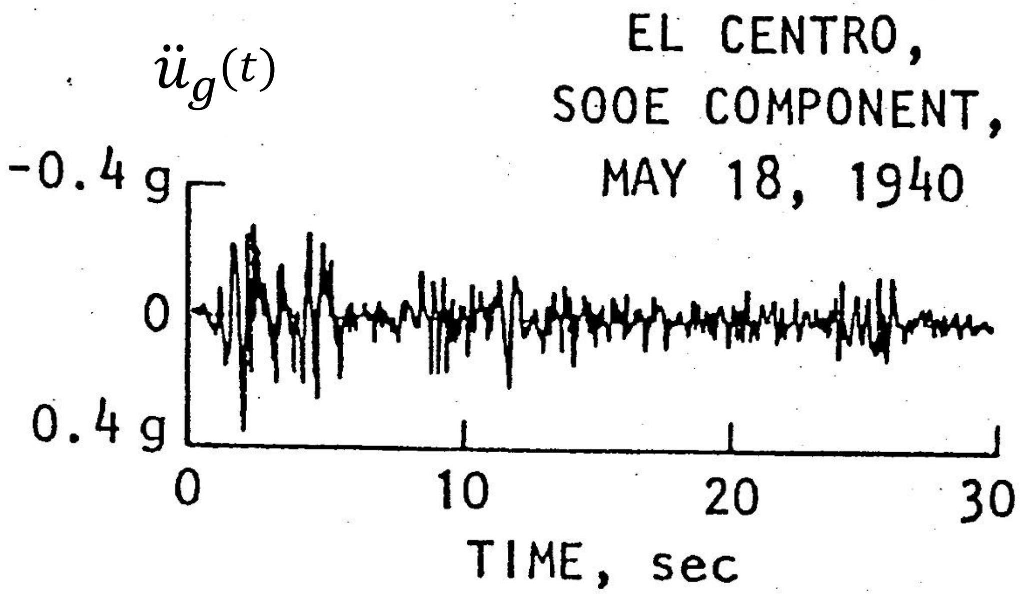
Evaluation of Structural Response to Earthquake Ground Motions

$$m \frac{d^2u(t)}{dt^2} + c\frac{du(t)}{dt} + k\,u(t) = -m\frac{d^2u_g(t)}{dt^2}$$

m

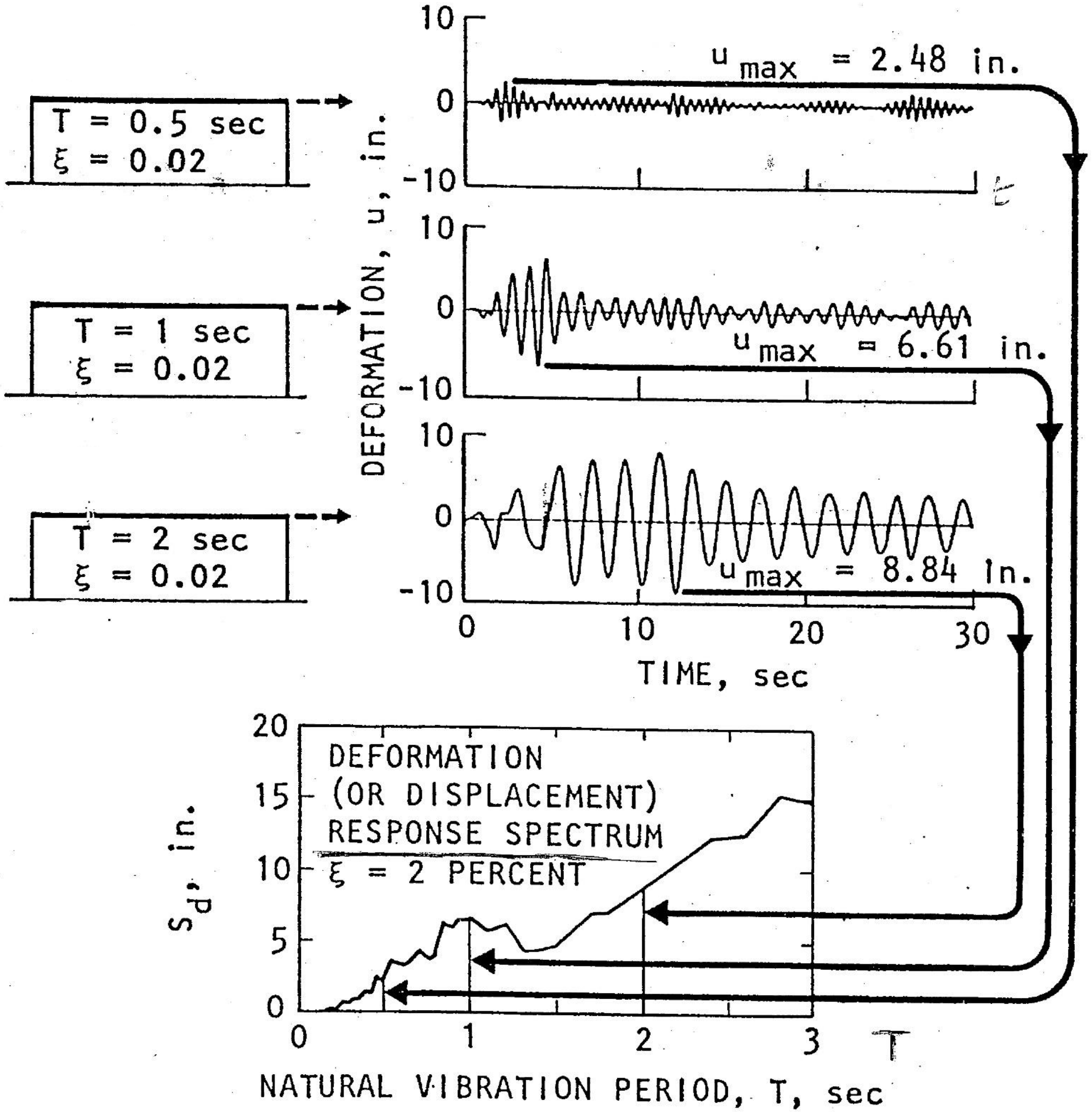


$\longleftrightarrow \ddot{u}_g(t)$



Applications

Evaluation of Response Spectrum of Earthquake Ground Motions



Computation of deformation (or displacement) response spectrum

Step-by-step Direct Integration Method

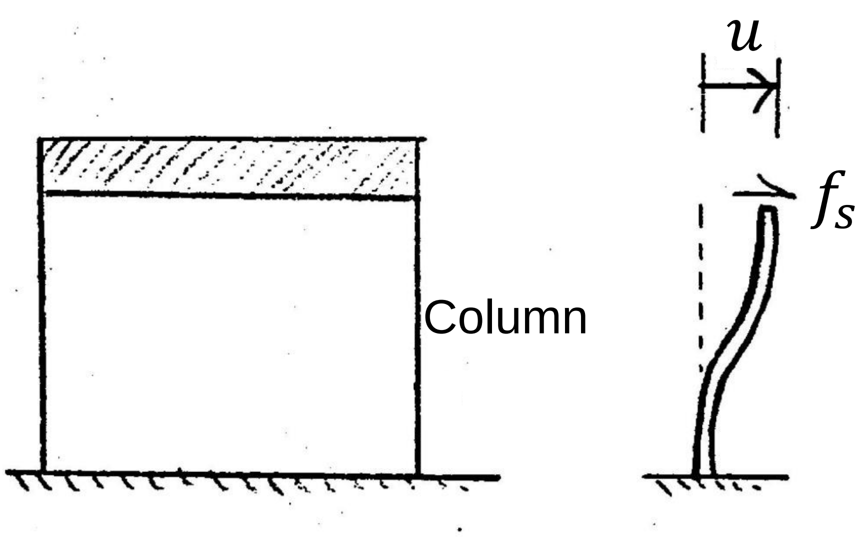
Or Time-stepping Method

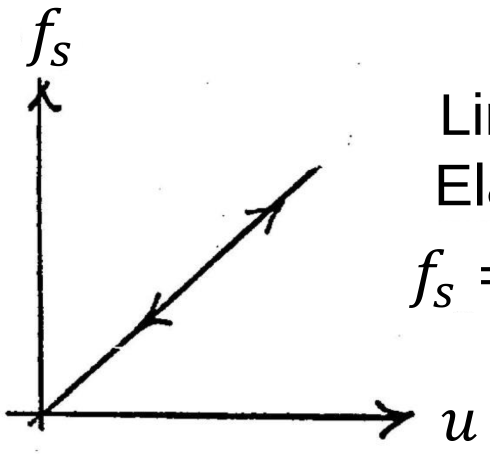
Step-by-step Direct Integration Method

- General Dynamic loadings
- Linear & Nonlinear Structures

- In some important structural dynamic problems, the responses of structures are in nonlinear range.

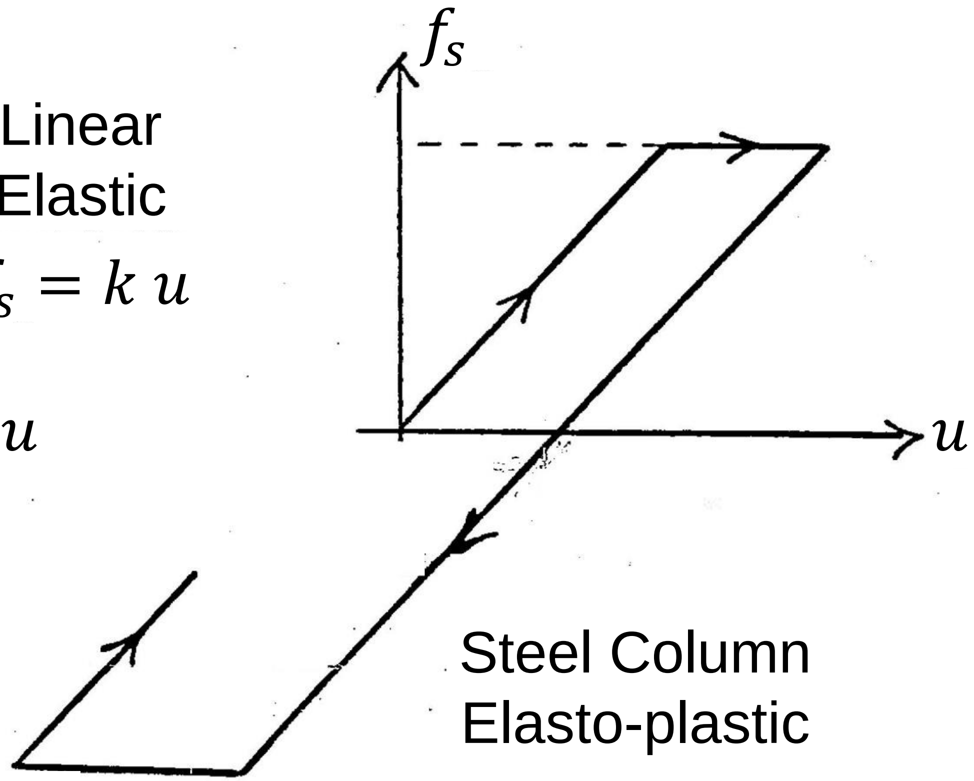
  For example, **the response of a structure subjected to a major earthquake**.



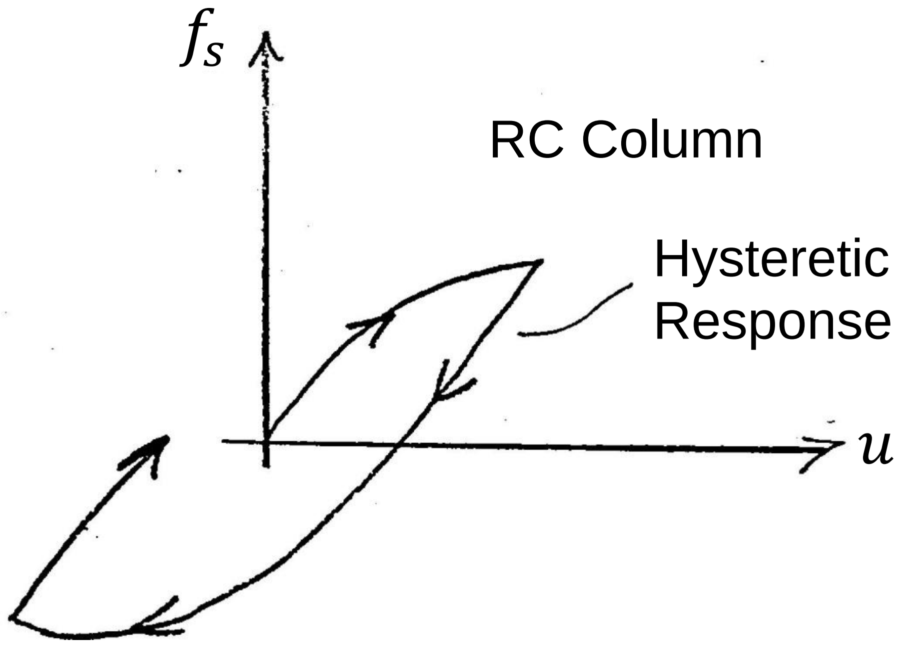


Linear Elastic

$f_s = k\,u$



Steel Column
Elasto-plastic



RC Column

Hysteretic Response { Concrete Cracking; Rebar Yielding }

Nonlinear System
Nonlinear Equation

→

Duhamel's Integral is not applicable

Step-by-step Integration Procedure

Consider the dynamic equilibrium (in scalar form) of a nonlinear structure at time t:

$$f_I(t) + f_D(t) + f_s(t) = p(t) \quad \text{.......... (1)}$$

Where

$$f_I(t) = m\,\ddot{u}(t)$$

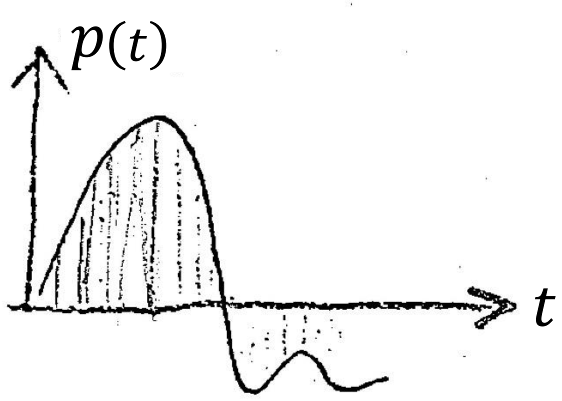
$$f_D(t) \neq c\dot{u}(t)$$

Damping force may not be a linear function of velocity $\dot{u}(t)$.

$$f_s(t) \neq k\,u(t)$$

Restoring force is a nonlinear function of displacement $u(t)$.

$p(t)$ is an arbitrary/general loading.



Step-by-step Integration Procedure

$$f_I(t) + f_D(t) + f_S(t) = p(t) \quad \text{.......... (1)}$$

At a small time Δt later:

$$f_I(t+\Delta t) + f_D(t+\Delta t) + f_S(t+\Delta t) = p(t+\Delta t) \quad \text{.......... (2)}$$

Subtract Equation (2) by Equation (1), we get

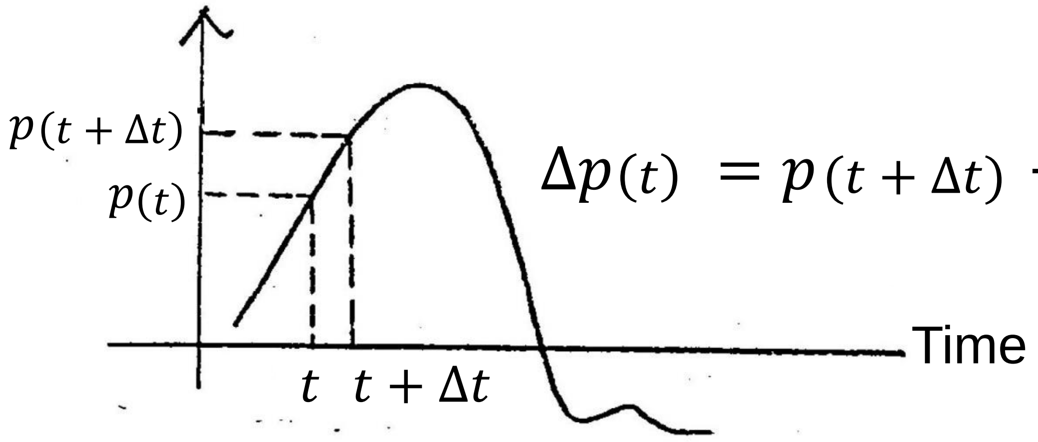
$$\Delta f_I(t) + \Delta f_D(t) + \Delta f_S(t) = \Delta p(t) \quad \text{.......... (3)}$$

Where

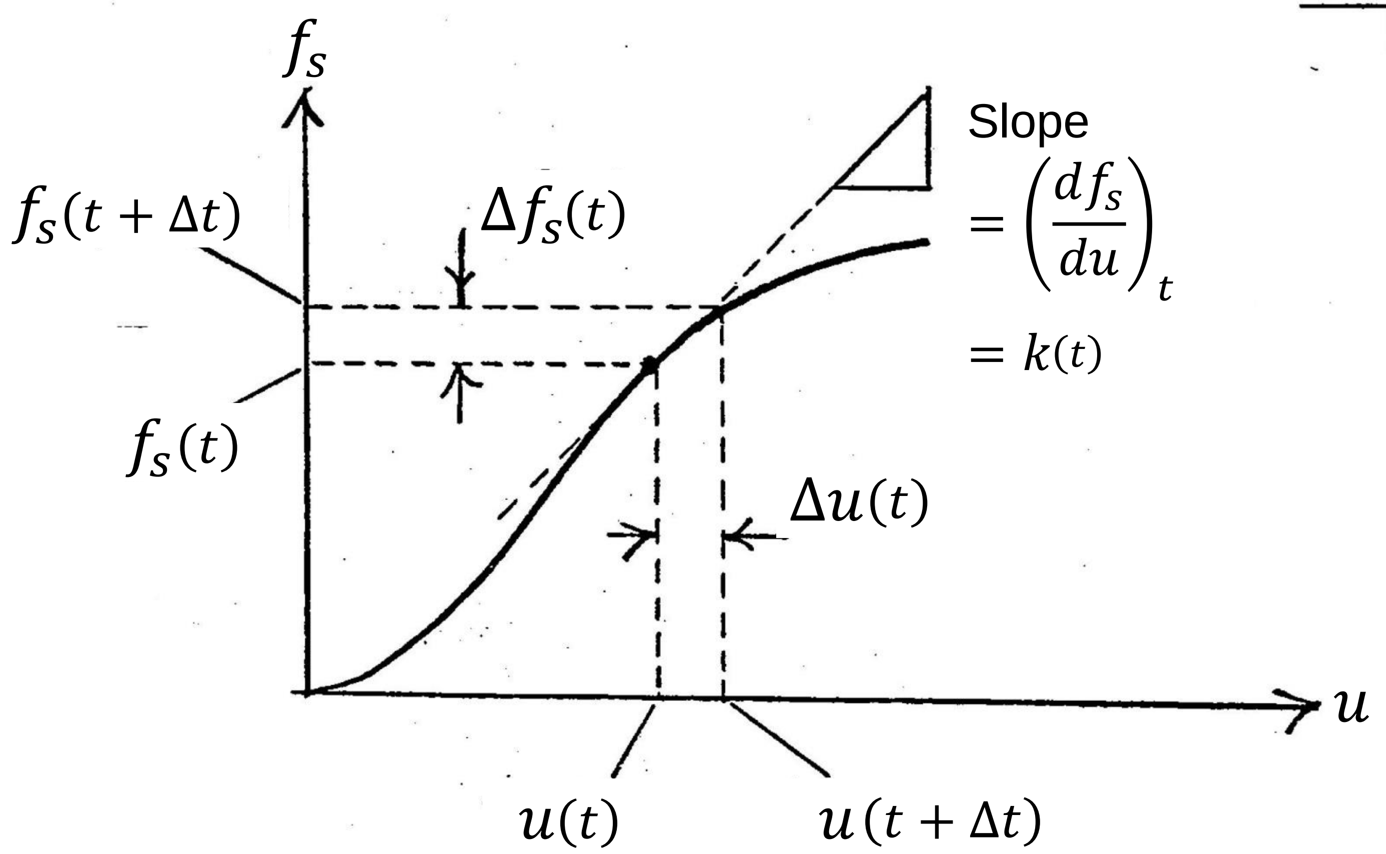
$$\Delta f_I(t) = f_I(t+\Delta t) - f_I(t) = m\,\Delta\ddot{u}(t)$$

$$\Delta f_D(t) = f_D(t+\Delta t) - f_D(t) \cong \left(\frac{df_D}{d\dot{u}}\right)_t . \Delta\dot{u}(t) = c(t).\Delta\dot{u}(t)$$

$$\Delta f_S(t) = f_S(t+\Delta t) - f_S(t) \cong \left(\frac{df_S}{du}\right)_t . \Delta u(t) = k(t).\Delta u(t)$$



$$\Delta p(t) = p(t+\Delta t) - p(t)$$



$$\Delta f_s(t) = f_s(t+\Delta t) - f_s(t)$$

$$\cong \left(\frac{df_s}{du}\right)_t . \Delta u(t)$$

$$\cong k(t) . \Delta u$$

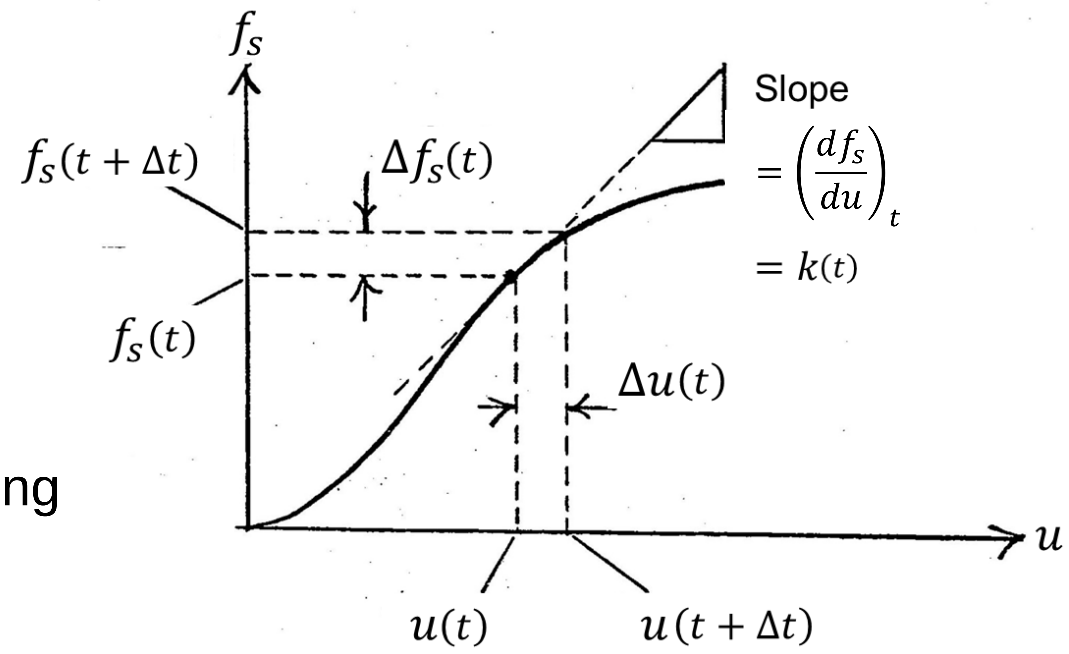
Step-by-step Integration Procedure

We have introduced the following two approximations:

$$\Delta f_D(t) = c(t) \, . \, \Delta\dot{u}(t)$$

$$\Delta f_s(t) = k(t) \, . \, \Delta u(t)$$

They are equivalent to the assumption that the damping and restoring forces are linear within t and $t + \Delta t$.



"Piecewise Linear Approximation of Structural System"

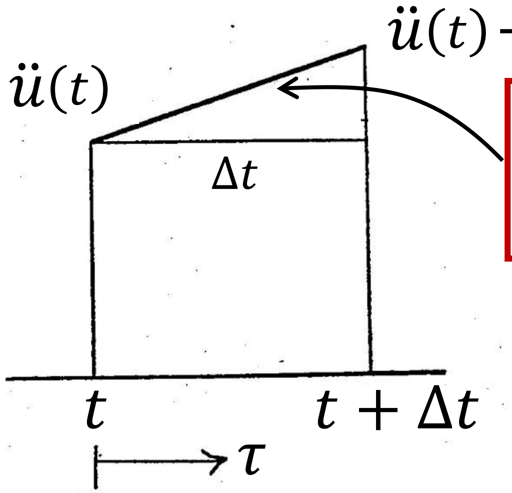
Step-by-step Integration Procedure

The incremental equation of motion (Equation 3) becomes,

$$m\,\Delta\ddot{u}(t) + c(t)\,\Delta\dot{u}\,(t) + k(t)\,\Delta u\,(t) = \Delta p\,(t) \qquad \text{……….} \; (4)$$

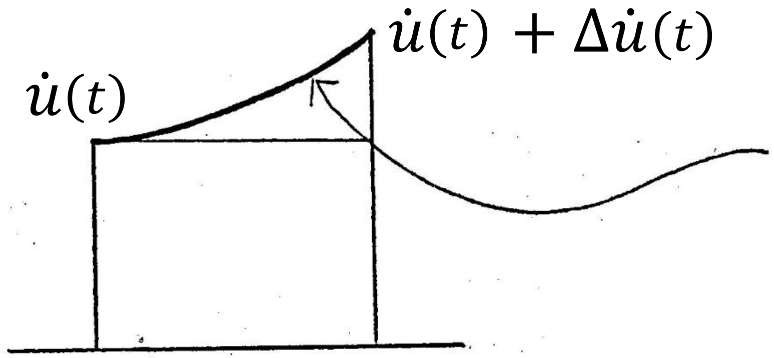
Introducing an assumption that "**the acceleration response varies linearly during each time increment**".

This yields quadratic and cubic variations of velocity and displacement, respectively.



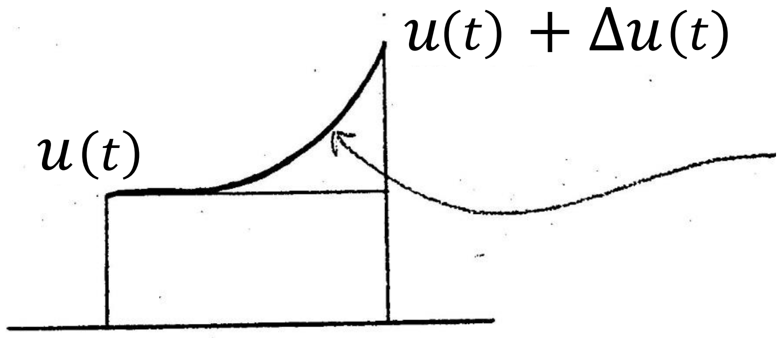
$$\ddot{u}(\tau) = \ddot{u}(t) + \frac{\Delta\ddot{u}(t)}{\Delta t}.\tau \quad \text{......... (5)}$$

"the acceleration response varies linearly during each time increment".



$$\dot{u}(\tau) = \dot{u}(t) + \int_0^{\tau} \ddot{u}(\tau)\, d\tau$$

$$\dot{u}(\tau) = \dot{u}(t) + \underbrace{\ddot{u}(t).\tau + \frac{\Delta\ddot{u}(t)}{\Delta t}.\frac{\tau^2}{2}}_{\Delta\dot{u}(\tau)} \quad \text{......... (6)}$$



$$u(\tau) = u(t) + \int_0^{\tau} \dot{u}(\tau)\, d\tau$$

$$u(\tau) = u(t) + \underbrace{\dot{u}(t).\tau + \ddot{u}(t).\frac{\tau^2}{2} + \frac{\Delta\ddot{u}(t)}{\Delta t}.\frac{\tau^3}{6}}_{\Delta u(\tau)} \quad (7)$$

Step-by-step Integration Procedure

At $\tau = \Delta t$, the above equations for velocity and displacement becomes,

$$\Delta\dot{u}(t) = \ddot{u}(t).\Delta t + \frac{\Delta\ddot{u}(t)}{\Delta t}.\frac{\Delta t^2}{2} \qquad \text{.......... (8)}$$

$$\Delta u(t) = \dot{u}(t).\Delta t + \ddot{u}(t).\frac{\Delta t^2}{2} + \frac{\Delta\ddot{u}(t)}{\Delta t}.\frac{\Delta t^3}{6} \qquad \text{.......... (9)}$$

Re-writing the above two equations in terms of $\Delta u(t)$:

$$\Delta\ddot{u}(t) = \frac{6}{\Delta t^2}.\Delta u(t) - \frac{6}{\Delta t}.\dot{u}(t) - 3\,\ddot{u}(t) \qquad \text{.......... (10)}$$

$$\Delta\dot{u}(t) = \frac{3}{\Delta t}.\Delta u(t) - 3\,\dot{u}(t) - \frac{\Delta t}{2}.\ddot{u}(t) \qquad \text{.......... (11)}$$

Equations (10) and (11) are derived from the **"linear acceleration assumption"**.

Step-by-step Integration Procedure

Introducing Equations (10) and (11) into the incremental form of governing equation of motion (Equation (4)), we obtain

$$m\left[\frac{6}{\Delta t^2}.\Delta u(t) - \frac{6}{\Delta t}.\dot{u}(t) - 3\,\ddot{u}(t)\right] + c(t)\left[\frac{3}{\Delta t}.\Delta u(t) - 3\,\dot{u}(t) - \frac{\Delta t}{2}.\ddot{u}(t)\right] + k(t).\Delta u(t) = \Delta p(t)$$

Re-writing the above equation, we get,

$$\tilde{k}(t)\,.\,\Delta u(t) = \Delta\,\tilde{p}(t) \qquad \text{..........} (12)$$

Where

$$\tilde{k}(t) = k(t) + \frac{6}{\Delta t^2}.m + \frac{3}{\Delta t}.c(t)$$

$$\Delta\tilde{p}_{(t)} = \Delta p_{(t)} + m\left[\frac{6}{\Delta t}.\dot{u}_{(t)} + 3\,\ddot{u}_{(t)}\right] + c_{(t)}\left[3\,\dot{u}_{(t)} + \frac{\Delta t}{2}.\ddot{u}_{(t)}\right]$$

Step-by-step Integration Procedure

$$\tilde{k}(t)\,.\,\Delta u(t) = \Delta\,\tilde{p}(t) \qquad \text{.......... (12)}$$

Where

$$\tilde{k}(t) = k(t) + \frac{6}{\Delta t^2}.m + \frac{3}{\Delta t}.c(t)$$

$$\Delta\tilde{p}(t) = \Delta p(t) + m\left[\frac{6}{\Delta t}.\dot{u}(t) + 3\,\ddot{u}(t)\right] + c(t)\left[3\,\dot{u}(t) + \frac{\Delta t}{2}.\ddot{u}(t)\right]$$

Let's assume that the calculation is made up to Time $= t$ and we are going to proceed to the next time stop, $t + \Delta t$.

Hence, $u(t)$, $\dot{u}(t)$, $\ddot{u}(t)$ are known, and $k(t)$, $c(t)$, m and $\Delta p(t)$ are also known.

$\Delta u(t)$ can be determined. $\Delta\dot{u}(t)$ and $\Delta\ddot{u}(t)$ can be derived from $\Delta u(t)$ by Eqs (11) and (10).

Step-by-step Integration Procedure

Note:

Two assumptions are used in this step-by-step calculation.

1) Within $\{t, t + \Delta t\}$, $\Delta f_D(t) = c(t).\Delta\dot{u}(t)$ and $\Delta f_s(t) = k(t).\Delta u(t)$

2) Within $\{t, t + \Delta t\}$, acceleration varies linearly

These assumptions are justified only when Δt is sufficiently small, small Δt → small error.

Although the error in each step is small, the error can accumulate and becomes significant when the number of steps is large.

The accumulation should be avoided by imposing the dynamic equilibrium condition at each time step.

Calculation flow chart

Time = t

$u(t)$ and $\dot{u}(t)$ are known

Evaluate $k(t) = \left(\frac{df_s}{du}\right)_t$, $c(t) = \left(\frac{df_D}{d\dot{u}}\right)_t$, $f_s(t) = f_s(u(t))$, $f_D(t) = f_D(\dot{u}(t))$

Impose dynamic equilibrium condition

$$\ddot{u}(t) = \frac{1}{m}.\left[p(t) - f_D(t) - f_s(t)\right] \qquad Equation\ (1)$$

$$\tilde{k}(t) = k(t) + \frac{6}{\Delta t^2}.m + \frac{3}{\Delta t}.c(t)$$

$$\Delta\tilde{p}(t) = \Delta p(t) + m\left[\frac{6}{\Delta t}.\dot{u}(t) + 3\,\ddot{u}(t)\right] + c(t)\left[3\,\dot{u}(t) + \frac{\Delta t}{2}.\ddot{u}(t)\right]$$

$$\Delta u(t) = \Delta\tilde{p}(t)/\tilde{k}(t)$$ Equation (12)

$$\Delta\dot{u}(t) = \frac{3}{\Delta t}\Delta u(t) - 3\dot{u}(t) - \frac{\Delta t}{2}\ddot{u}(t)$$ Equation (11)

Time = $t + \Delta t$

$$\dot{u}(t + \Delta t) = \dot{u}(t) + \Delta\dot{u}(t)$$

$$u(t + \Delta t) = u(t) + \Delta u(t)$$

Additional Notes

1. Response of any SDF system with any prescribed nonlinear properties can be evaluated by “step-by-step integration”.

2. Response of any linear SDF system can also be evaluated by the step-by-step integration.

3. To determine Δt, we should consider :

   - The rate of variation of the applied loading $p\ (t)$
   - The nonlinearity of damping and stiffness properties.
   - The natural period of structure (T)

Additional Notes

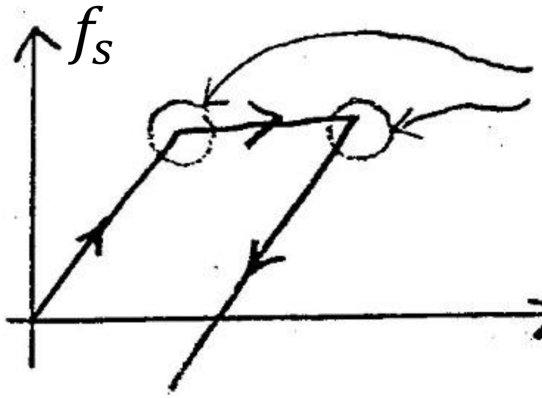
The choice of Δt also depends on the nonlinear properties of damping and stiffness

Rule of thumb:

$$\Delta t/T \ \le \ 1/10$$

My suggestion:

$$\Delta t/T \ \le \ 1/30$$



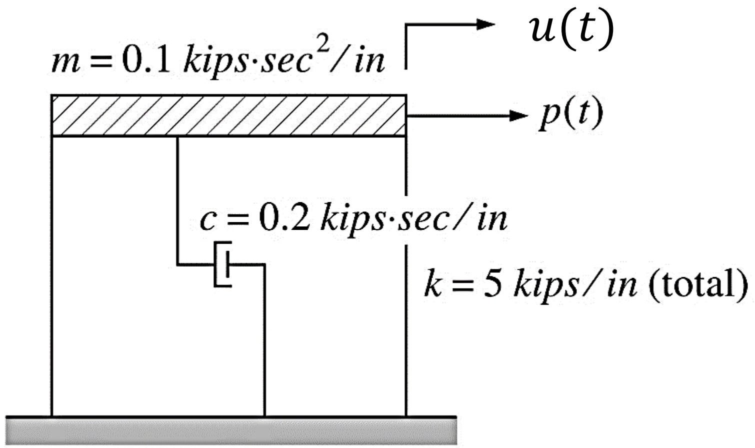
Need a very small Δt

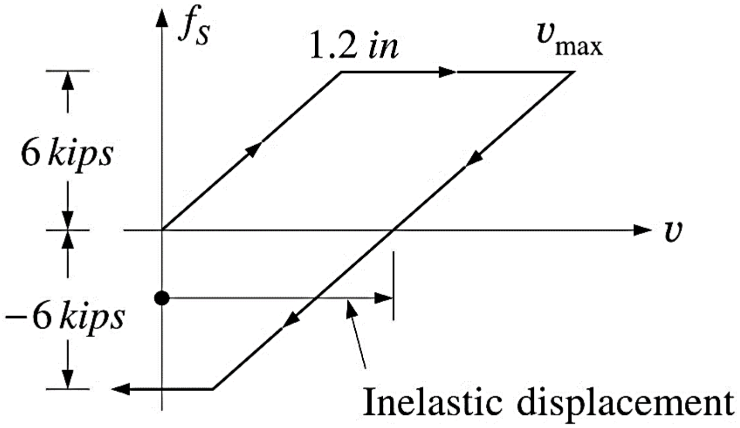
4. The step-by-step integration technique will be extended for the calculation of responses of nonlinear MDF systems later.

   More attention will be paid on the accumulation of error – as it is a major factor in the determination of Δt.

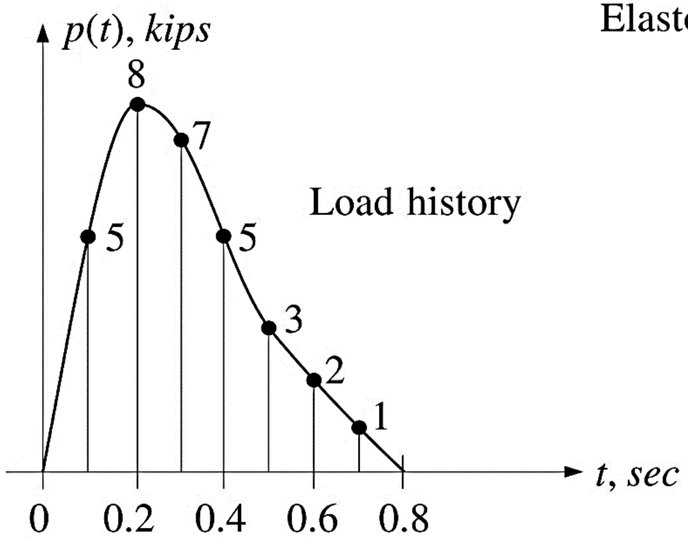
Numerical Example

Taken from Clough and Penzien (2003)





Elastoplastic stiffness



An elastoplastic frame and dynamic loading

Numerical Example Taken from Clough and Penzien (2003)

TABLE E7-1

Nonlinear response analysis: linear acceleration step-by-step method

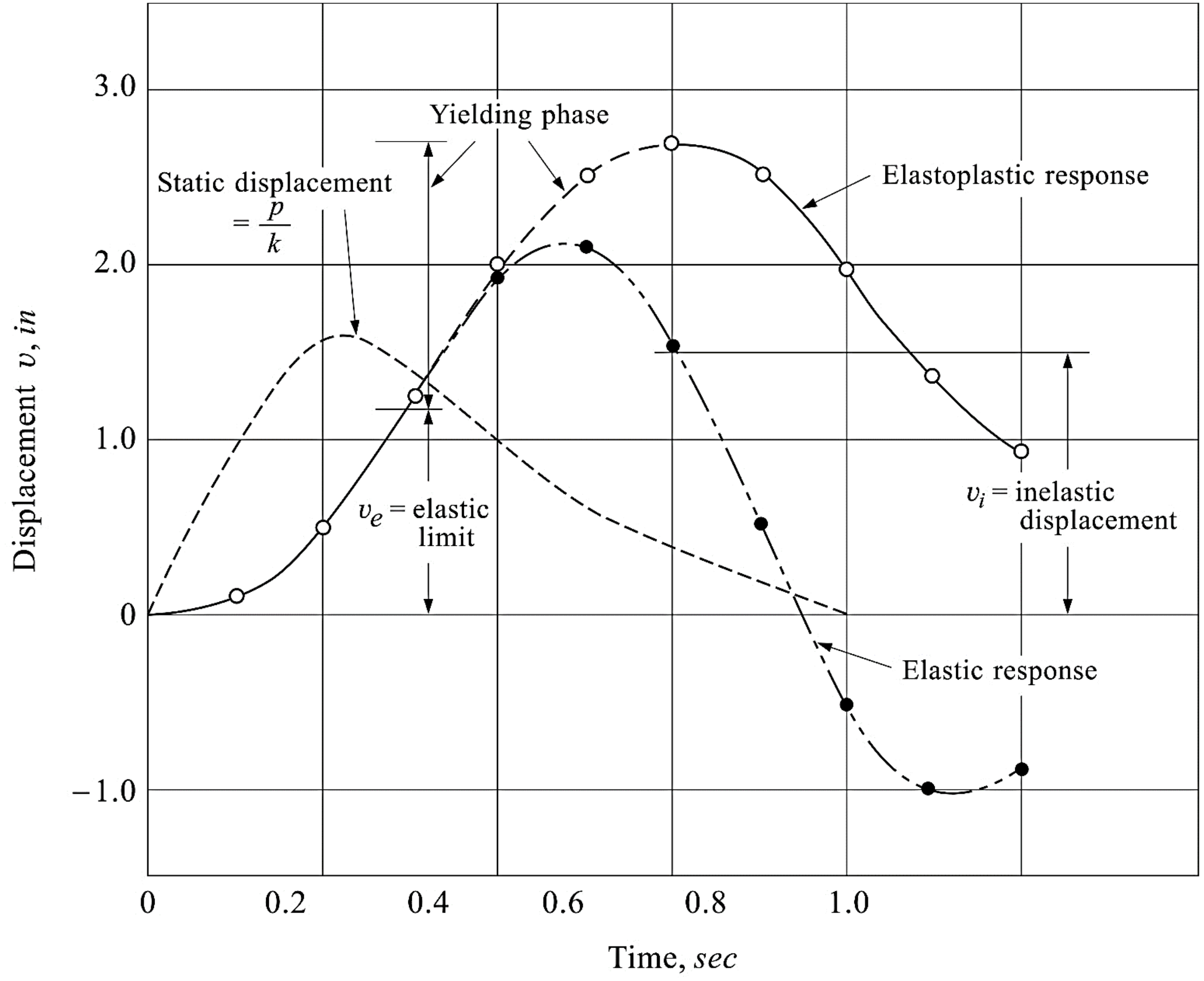
Structure and loading in Fig. E7-3

t	p	v	$\dot{v}$	f_S	f_D	f_I	$\ddot{v}$	Δp	$6.6\,\dot{v}$	$0.31\,\ddot{v}$	$\Delta \tilde{p}\,d$	k	$\tilde{k}$	Δ_v	$30\,\Delta_v$	$3\,\dot{v}$	$0.05\,\ddot{v}$	$\Delta_{\dot{v}}$
				$5\,\bar{v}$ *	$0.2\,\dot{v}$	(2)–(5)–(6)	10×(7)				(9)+(10)+(11)		66+(13)	(12)÷(14)				(16)–(17)–(18)
sec	kips	in	in/sec															
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
0.0	0	0	0	0	0	0	0	5	0	0	0	5	71	0.070	2.11	0	0	2.11
0.1	5	0.070	2.11	0.35	0.42	4.23	42.3	3	13.92	13.12	30.04	5	71	0.423	12.68	6.33	2.11	4.24
0.2	8	0.493	6.35	2.46	1.27	4.27	42.7	−1	41.90	13.25	54.15	5	71	0.763	22.88	19.06	2.14	1.68
0.3	7	1.256	8.03	6	1.61	−0.61	−6.1	−2	53.02	−1.89	49.13	0**	66	0.744	22.33	24.08	−0.30	−1.45
0.4	5	2.000	6.58	6	1.32	−2.32	−23.2	−2	43.43	−7.19	34.24	0	66	0.519	15.57	19.74	−1.16	−3.01
0.5	3	2.519	3.57	6	0.71	−3.71	−37.1	−1	23.56	−11.50	11.06	0	66	0.168	5.02	10.72	−1.85	−3.85
0.6	2	2.687	−0.28	6	−0.06	−3.94	−39.4	−1	−1.85	−12.22	−15.07	5	71	−0.212	−6.36	−0.84	−1.97	−3.55
0.7	1	2.475	−3.83	4.94	−0.77	−3.17	−31.7	−1	−25.28	−9.82	−36.10	5	71	−0.508	−15.24	−11.49	−1.58	−2.17
0.8	0	1.967	−6.00	2.40	−1.20	−1.20	−12.0	0	−39.60	−3.72	−43.32	5	71	−0.610	−18.30	−18.00	−0.60	0.30
0.9	0	1.357	−5.70	−0.65	−1.14	1.79	17.9	0	−37.62	5.55	−32.07	5	71	−0.452	−13.56	−17.10	0.90	2.64
1.0	0	0.905	−3.06															

* $\bar{v} = v - v_i$, where v_i = inelastic displacement = $v_{max} - 1.2$ *in*;

** $k = 0$ while frame is yielding.

Numerical Example Taken from Clough and Penzien (2003)



Comparison of elastoplastic with elastic response

Thank you