CE 809 - Structural Dynamics

Lecture 4: Response of SDF Systems to Periodic Loading

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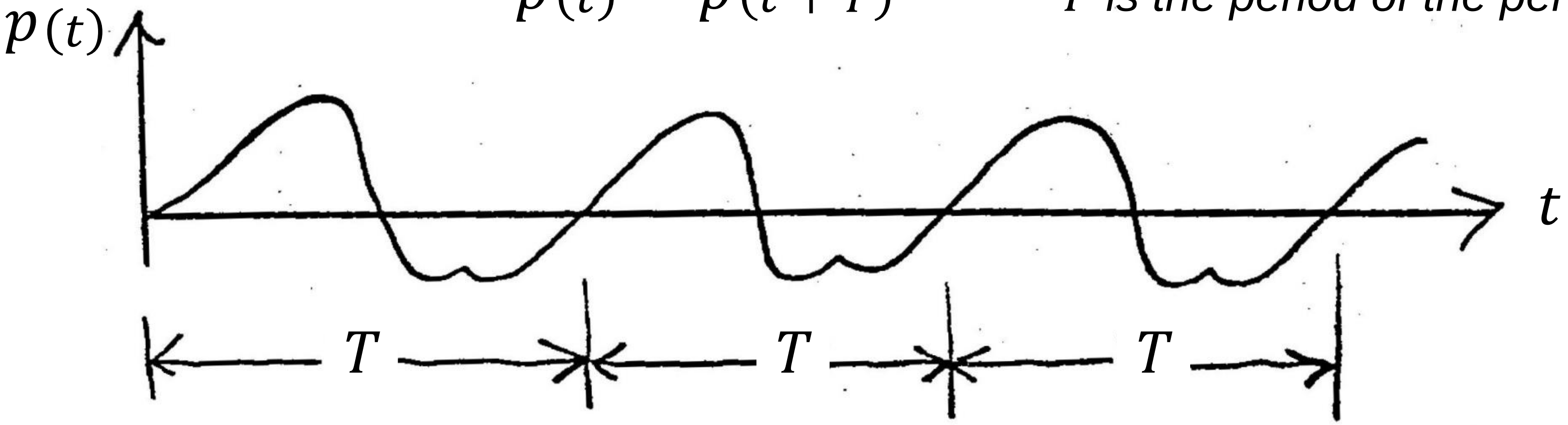
Periodic Loading

A SDF system is subjected to a "periodic force" $p(t)$

$$p(t) = p(t + T)$$

T is the period of the periodic force.



- A periodic function is one in which the portion defined over a time T **repeats itself indefinitely** as shown in the figure.

- Many forces are periodic or nearly periodic. For example, under certain conditions, propeller forces on a ship, wave loading on an offshore platform, and wind forces induced by vortex shedding on tall, slender structures are nearly periodic.

Fourier Series Representation of a Periodic Function

Any arbitrary periodic functions can be represented in terms of a summation of simple sine and cosine functions.

$$p_{(t)} = a_o + \sum_{n=1}^{\infty} a_n \cos(n\,\bar{\omega}\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,\bar{\omega}\,t) \qquad \text{..........} \ (1)$$

Where $\bar{\omega} = 2\pi/T$ and $a_o, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are called Fourier coefficients.

The right hand side of the above expression is called "**Fourier series**", i.e. a periodic function can be separated (decomposed) into its harmonic components in the Fourier series.

Fourier Decomposition

- This concept called **Fourier decomposition** was first proposed by Jean-Baptiste Joseph Fourier, a French physicist and mathematician (1768 - 1830).

- The beginnings on Fourier series can also be found in works by Leonhard Euler and by Daniel Bernoulli, but it was Fourier who employed them in a systematic and general manner in his main work, “Théorie analytique de la chaleur (Analytic Theory of Heat, Paris, 1822)”.

- It is a very powerful mathematical concept.

Refer to “Advanced Engineering Mathematics” by Erwin Kreszig, 10th Edition).

Joseph Fourier (1768 - 1830)

Fourier Series

If $p(t)$ is given, the coefficients a_n and b_n can be determined by simple integrations as follows.

$$\int_{t=0}^{t=T} p(t)\, dt = \int_{t=0}^{t=T} \left[a_o + \sum_{n=1}^{\infty} a_n \cos(n\,\bar{\omega}\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,\bar{\omega}\,t) \right] dt = a_o\, T$$

$$a_o = \frac{1}{T} \int_{t=0}^{t=T} p(t)\, dt \qquad \text{..........} \; (2)$$

Fourier Series

$$\int_{t=0}^{t=T} p(t)\cos(m\,\bar{\omega}\,t)\,dt$$

$$= \int_{t=0}^{t=T} \left[a_o + \sum_{n=1}^{\infty} a_n \cos(n\,\bar{\omega}\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,\bar{\omega}\,t)\right] \cos(m\,\bar{\omega}\,t)\,dt = \frac{a_m\,T}{2}$$

Therefore,

$$a_m = \frac{2}{T}\int_{t=0}^{t=T} p(t)\cos(m\,\bar{\omega}\,t)\,dt \qquad \text{..........}\ (3)$$

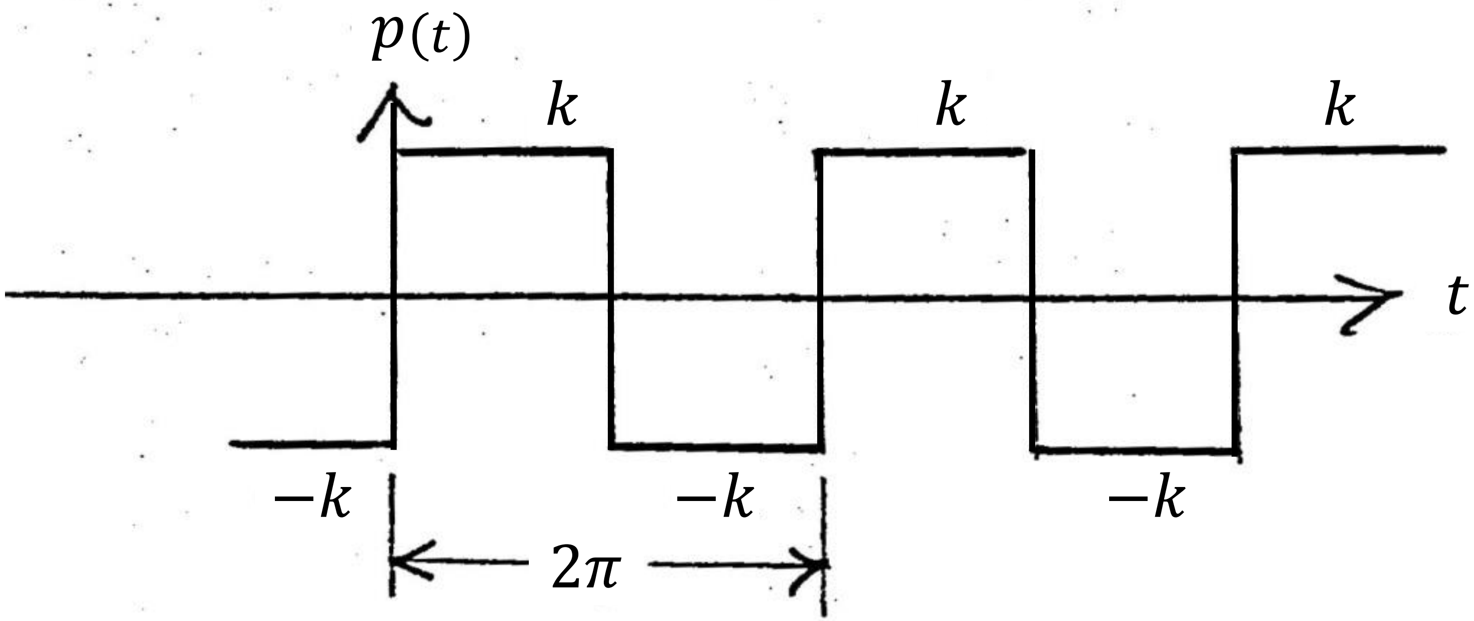
Similarly, it can be shown that,

$$b_m = \frac{2}{T}\int_{t=0}^{t=T} p(t)\sin(m\,\bar{\omega}\,t)\,dt \qquad \text{..........}\ (4)$$

Example

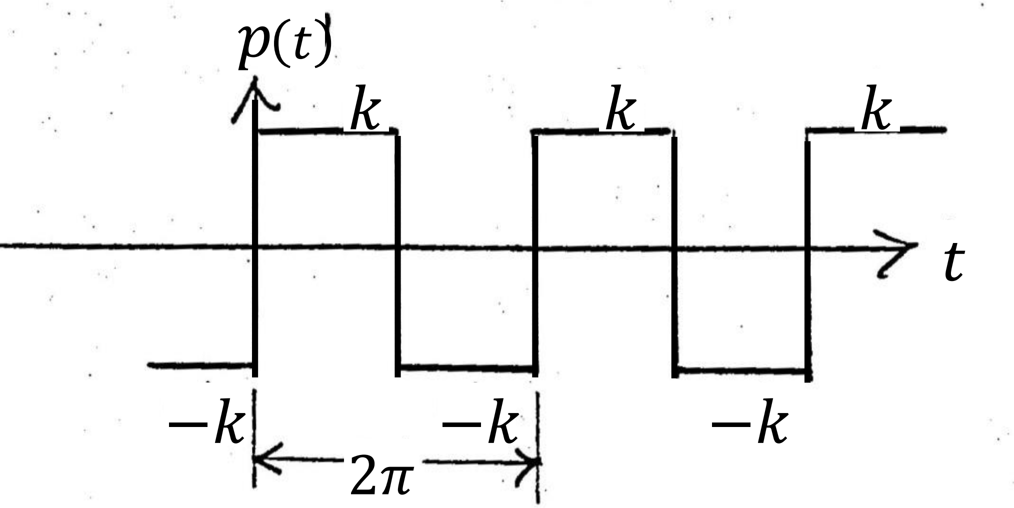
Consider a periodic square function as shown below.

$$p(t) = \begin{cases} k & for\ 0 < t < \pi \\ -k & for\ \pi < t < 2\pi \end{cases}$$



Example

Conducting the integrations as shown be the equations (2), (3) and (4), we obtain,



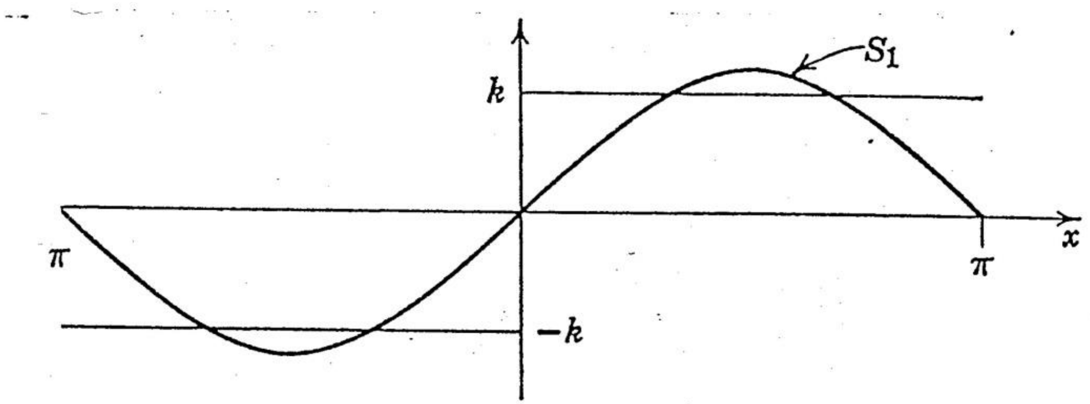
$$a_o = 0$$

$$a_n = 0$$

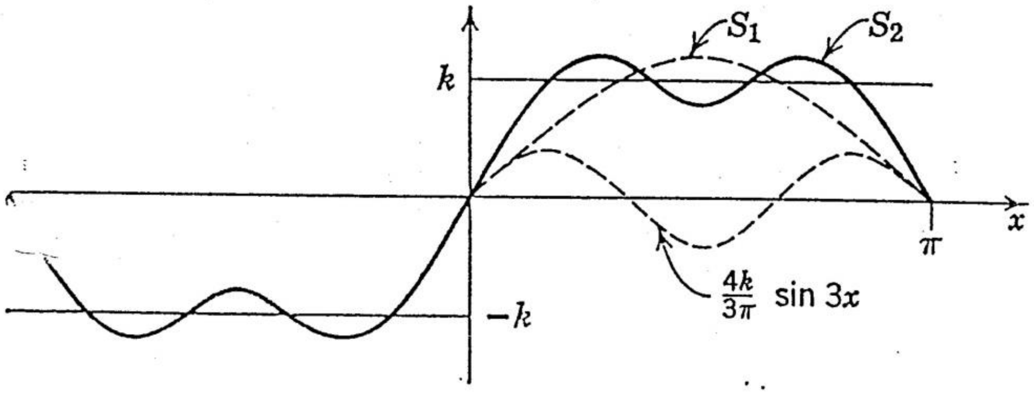
$$b_n = \frac{2\,k}{n\,\pi}(1 - \cos(n\,\pi)) \qquad n = 1, 2, 3, \ldots\ \infty$$

That is,

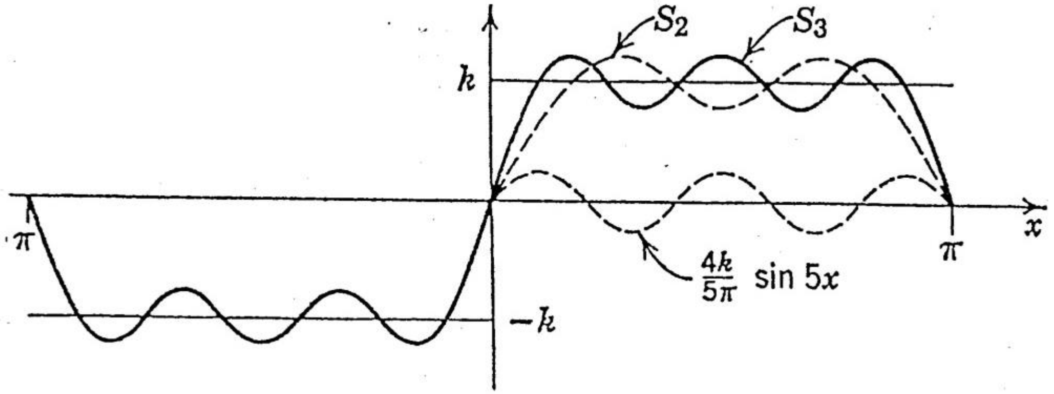
$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \ldots$$



$$S_1 = b_1 \sin(\overline{\omega}\, t)$$



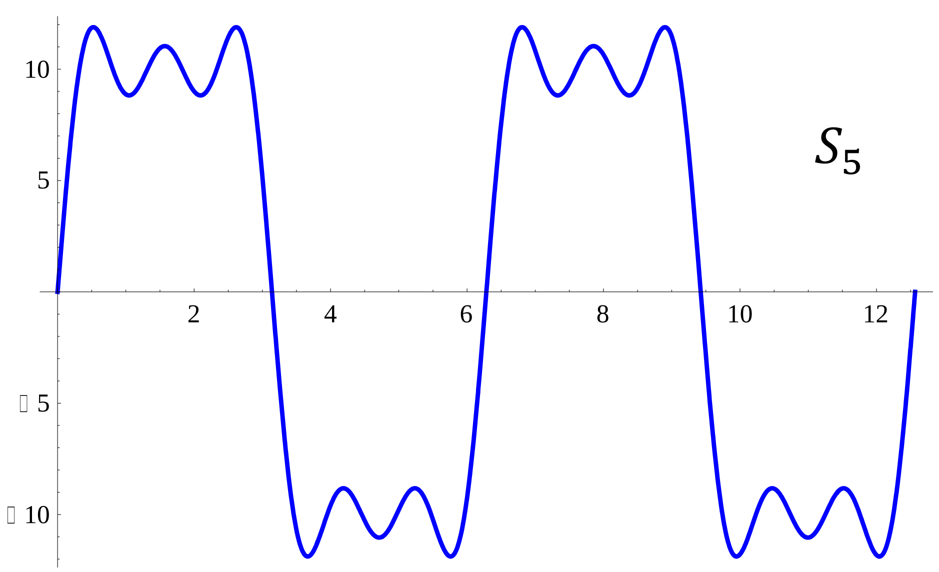
$$S_3 = b_1 \sin(\overline{\omega}\, t) + b_3 \sin(3\, \overline{\omega}\, t)$$

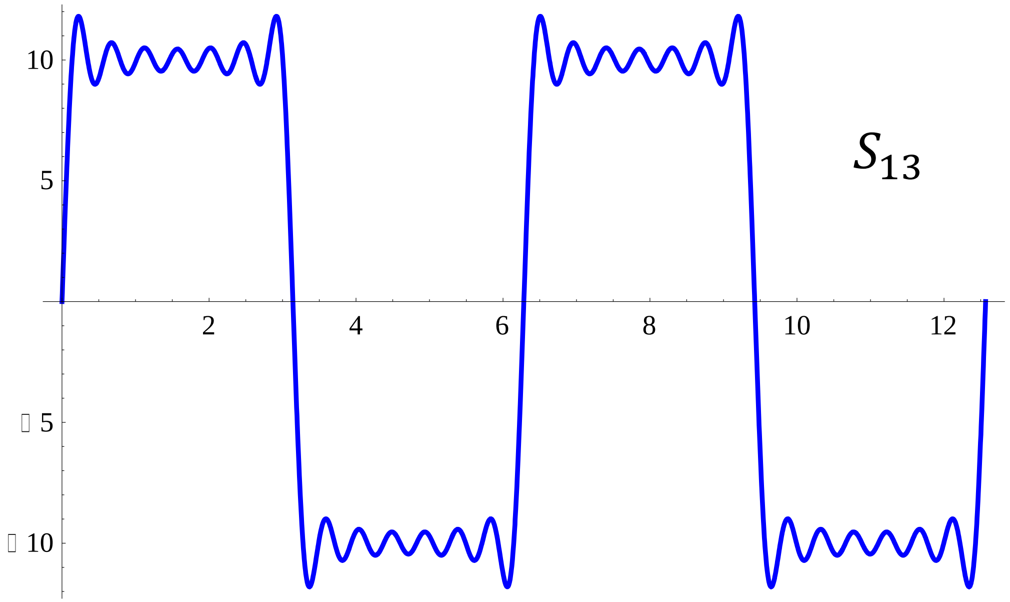


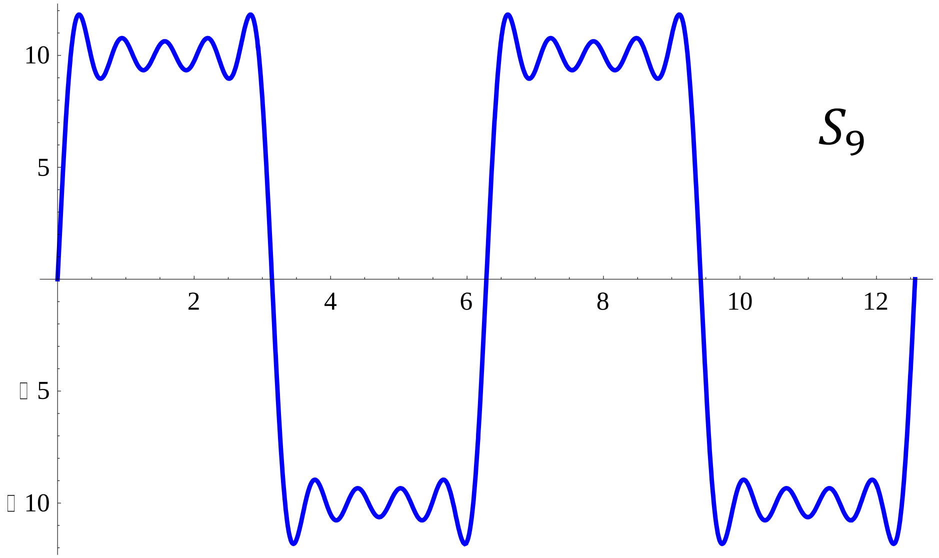
$$S_5 = b_1 \sin(\overline{\omega}\, t) + b_3 \sin(3\, \overline{\omega}\, t) + b_5 \sin(5\, \overline{\omega}\, t)$$

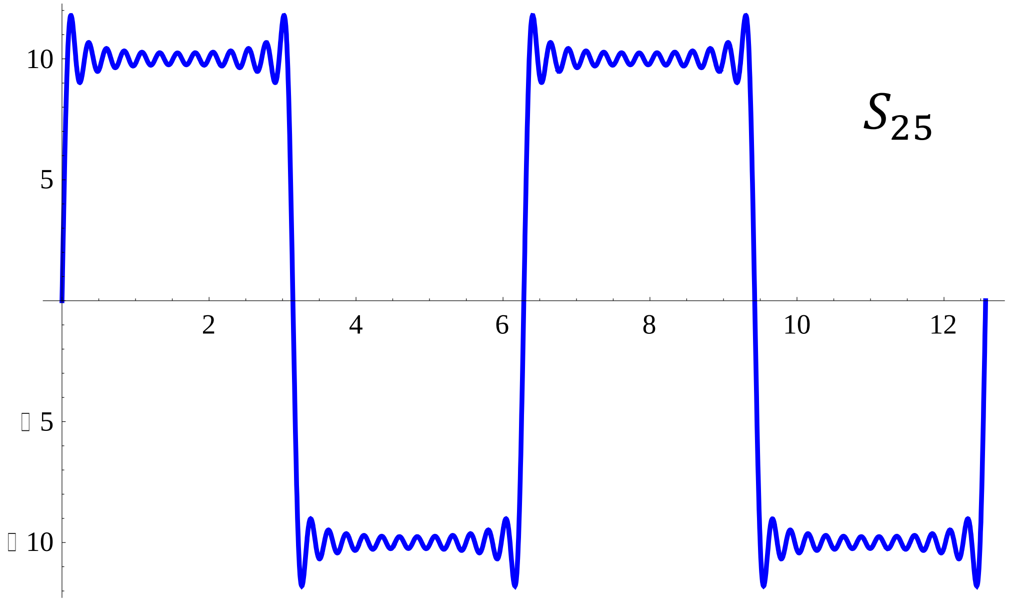
The first three partial sums of the corresponding

Fourier series of the given square periodic function


10
5
−5
−10
2
4
6
8
10
12
S_5


10
5
−5
−10
2
4
6
8
10
12
S_{13}


10
5
−5
−10
2
4
6
8
10
12
S_9


10
5
−5
−10
2
4
6
8
10
12
S_{25}

Example

The series **coverage quickly to the square function**.

Theoretically, an infinite number of terms are required for the Fourier series to converge to $p(t)$.

In practice, however, a few terms are sufficient for good convergence.

Therefore, in many practical applications, it is not necessary to evaluate ∞ series. Only a finite series is good enough.

$$p(t) \cong \sum_{n=1}^{N} b_n \sin(n\,\bar{\omega}\,t)$$

Where N is finite, not ∞

Response to a Periodic Loading

Response to a periodic loading = Response to the Fourier series of the loading

Superposition ↓

= the sum of the responses to each sine and cosine loadings in the series

Response to a Periodic Loading

Superposition

Let $u_1(t)$ be response to $p_1(t)$ loading i.e.

$$m\,\ddot{u}_1(t) + c\,\dot{u}_1(t) + k\,u_1(t) = p_1(t)$$

And $u_2(t)$ be the response to $p_2(t)$ i.e.

$$m\,\ddot{u}_2(t) + c\,\dot{u}_2(t) + k\,u_2(t) = p_2(t)$$

Then $u_1(t) + u_2(t)$ is the response to $p_1(t) + p_2(t)$.

$$m(\ddot{u}_1(t) + \ddot{u}_2(t)) + c(\dot{u}_1(t) + \dot{u}_2(t)) + k(u_1(t) + u_2(t)) = p_1(t) + p_2(t)$$

Steady-state Response to a Periodic Loading

$$p(t) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\,\bar{\omega}\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,\bar{\omega}\,t)$$

$$u_{oa} = \frac{a_o}{k}$$

Define $\beta_n = n\,\bar{\omega}/\omega$ and use the result obtained from the previous section.

$u_{bn}(t)$ = steady-state response to $b_n \sin(n\bar{\omega}t)$

$$u_{bn}(t) = \frac{b_n}{k} \frac{1}{\left(1-{\beta_n}^2\right)^2 + (2\,\xi\,\beta_n)^2} \left\{\left(1-{\beta_n}^2\right)\sin(n\,\bar{\omega}\,t) - 2\,\xi\,\beta_n \cos(n\,\bar{\omega}\,t)\right\}$$

Steady-state Response to a Periodic Loading

$$u_{bn}(t) = \frac{b_n}{k} \frac{1}{\left(1 - {\beta_n}^2\right)^2 + (2\,\xi\,\beta_n)^2} \left\{\left(1 - {\beta_n}^2\right) \sin(n\,\overline{\omega}\,t) - 2\,\xi\,\beta_n \cos(n\,\overline{\omega}\,t)\right\}$$

$$u_{an}(t) = \frac{a_n}{k} \frac{1}{\left(1 - {\beta_n}^2\right)^2 + (2\,\xi\,\beta_n)^2} \left\{2\,\xi\,\beta_n \sin(n\,\overline{\omega}\,t) + \left(1 - {\beta_n}^2\right) \cos(n\,\overline{\omega}\,t)\right\}$$

Steady-state Response to a Periodic Loading

The combined response would be,

$$u(t) = \frac{1}{k}\left[a^0 + \sum_{n=1}^{\infty} \frac{1}{\left(1-{\beta_n}^2\right)^2 + (2\,\xi\,\beta_n)^2}\left\{\left(a_n\,2\,\xi\,\beta_n + b_n\left(1-{\beta_n}^2\right)\right)\sin(n\,\overline{\omega}\,t) + \left(a_n\left(1-{\beta_n}^2\right) - b_n\,2\,\xi\,\beta_n\right)\cos(n\,\overline{\omega}\,t)\right\}\right]$$

Example

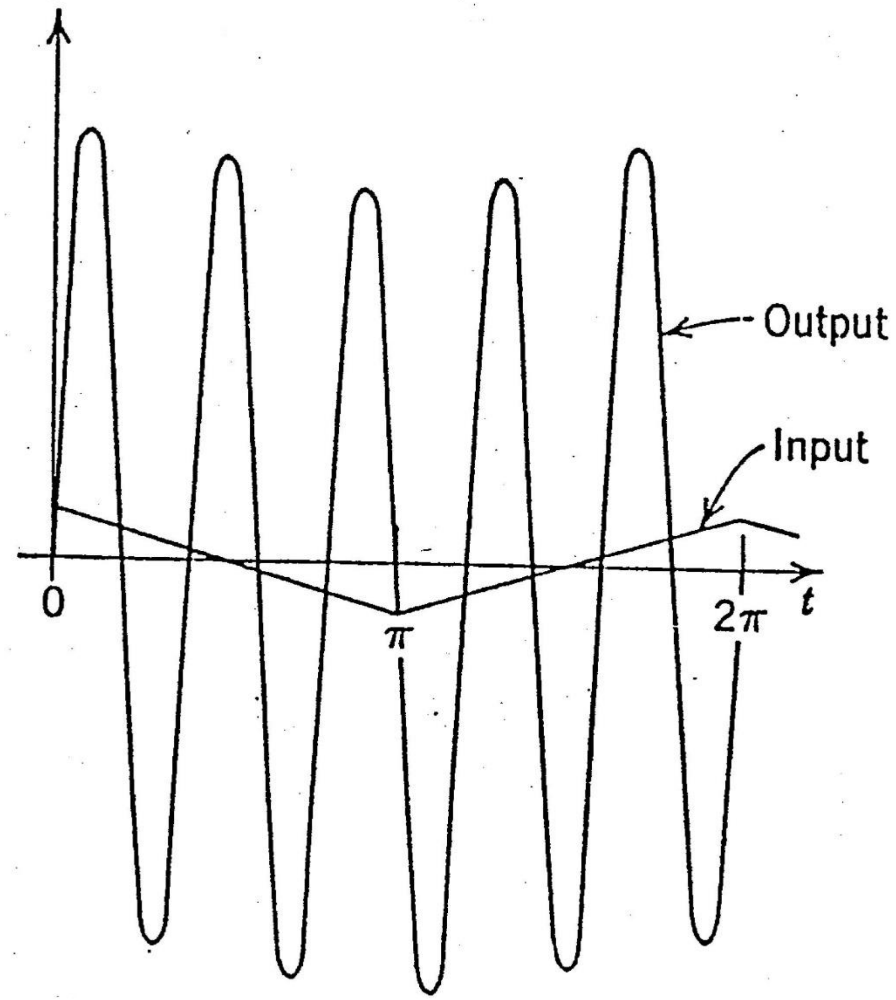
Response of an SDF structure with $\omega = 5$ rad/sec when subjected to a periodic loading of triangular waveform ($\bar{\omega} = 1$ rad/sec)

Inputs: $\bar{\omega} = 1, \qquad \omega = 5$ rad/sec

Fourier Series:

$$\beta_1 = \frac{\bar{\omega}}{\omega} = 0.2, \quad \beta_3 = 3\frac{\bar{\omega}}{\omega} = 0.6, \quad \beta_5 = 5\frac{\bar{\omega}}{\omega} = 1, \quad \ldots$$

For β_5 term, the response will be dominated by the **resonance response** at frequency $5\,\bar{\omega}$.



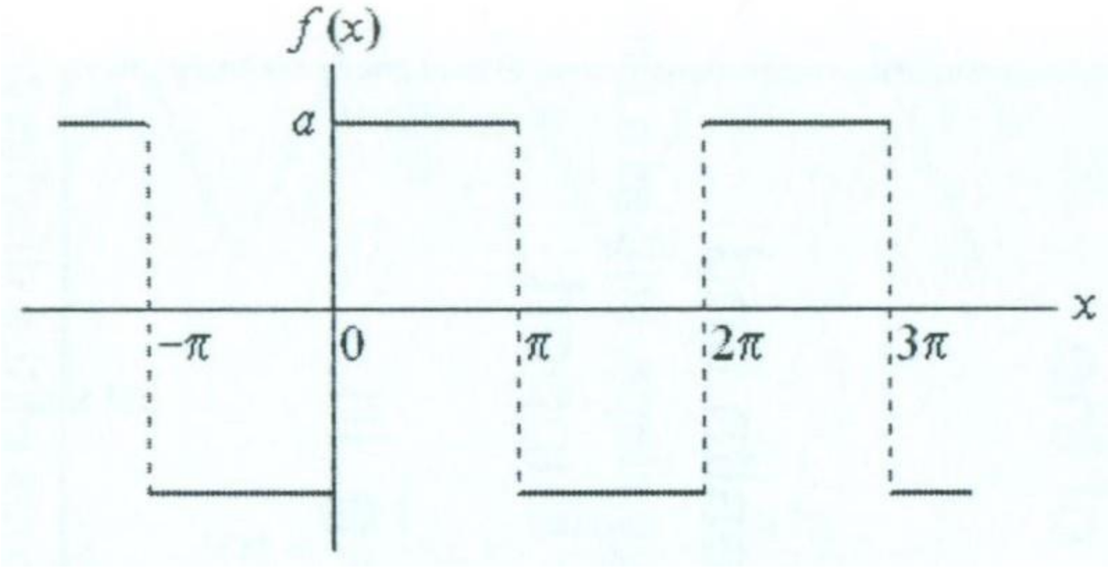
An example steady state response of an input triangular force

Appendix

Fourier Series of some Common Periodic Functions

Function:

$$f(x) = \begin{cases} a & for\ 0 < x < \pi \\ -a & for\ -\pi < x < 0 \end{cases}$$

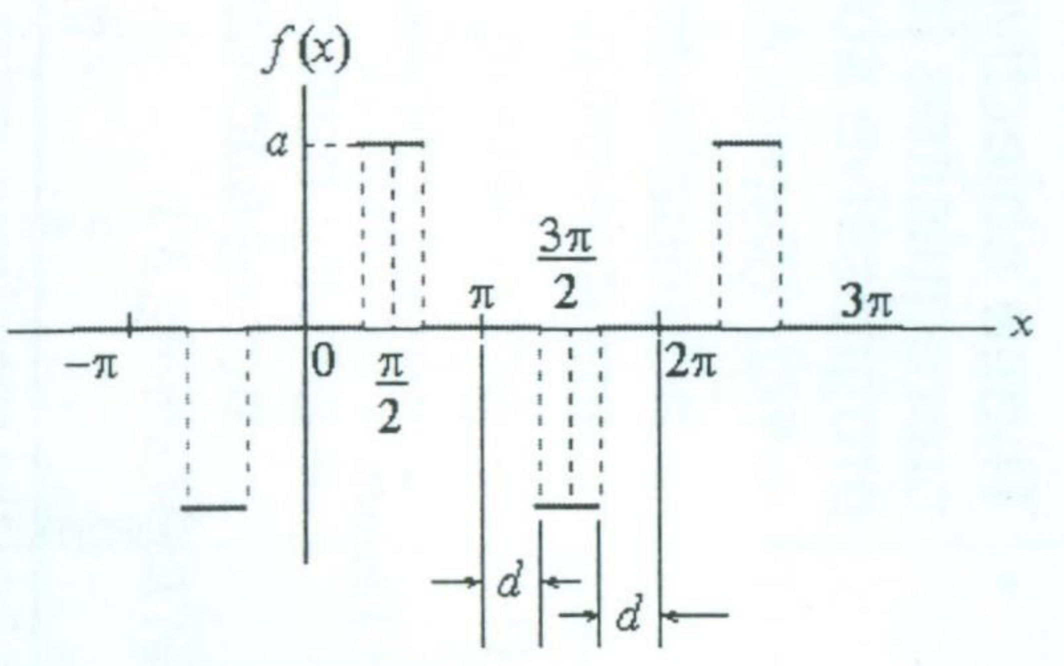


Fourier series:

$$f(x) = \frac{4a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Function:

$$f(x) = \begin{cases} a & for\, d < x < \pi - d \\ -a & for\, \pi + d < x < 2\pi - d \end{cases}$$

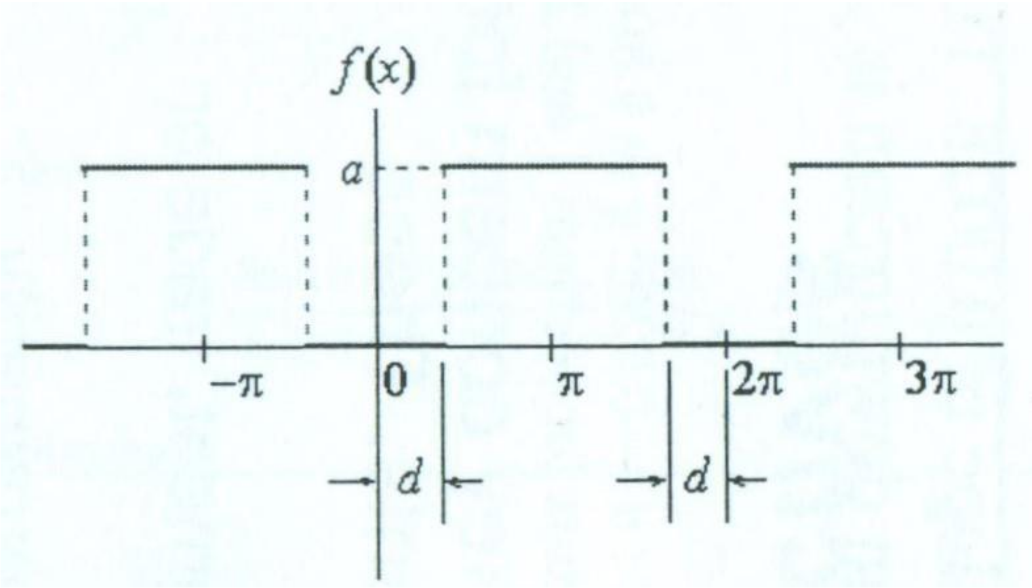


Fourier series:

$$f(x) = \frac{4a}{\pi}\left(\cos d \sin x + \frac{1}{3}\cos 3d \sin 3x + \frac{1}{5}\cos 5d \sin 5x + \cdots\right)$$

Function:

$$f(x) = \begin{cases} a & for\ d < x < 2\pi - d \\ 0 & for\ 0 < x < d\ , 2\pi - d < x < 2\pi \end{cases}$$

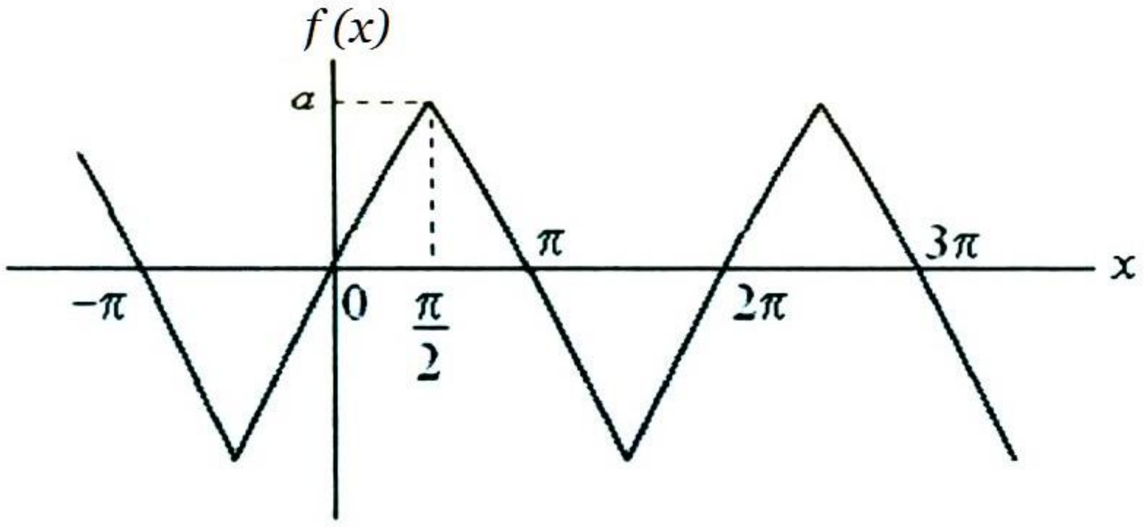


Fourier series:

$$f(x) = \frac{2a}{\pi}\left(\frac{\pi - d}{2} - \frac{\sin(\pi - d)}{1}\cos x + \frac{\sin 2(\pi - d)}{2}\cos 2x - \frac{\sin 3(\pi - d)}{3} + \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{2ax}{\pi} & for -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\ \dfrac{2a(\pi - x)}{\pi} & for\ \dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2} \end{cases}$$

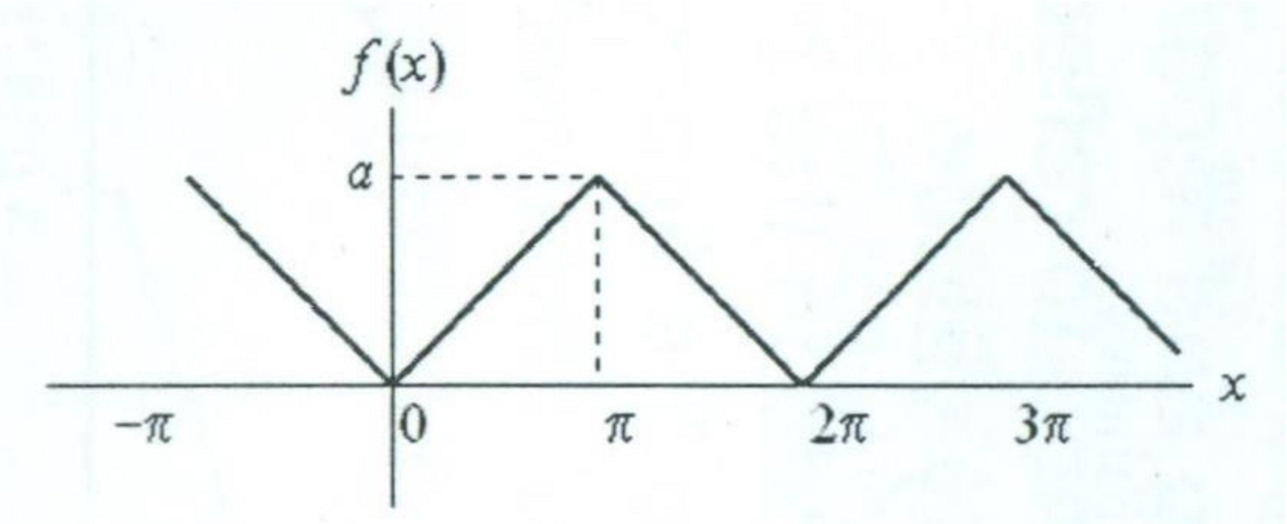


Fourier series:

$$f(x) = \frac{8a}{\pi^2}\left(\frac{\sin x}{1} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{\pi} & for\ 0 \le x \le \pi \\ \dfrac{a(2\pi - x)}{\pi} & for\ \pi \le x \le 2\pi \end{cases}$$

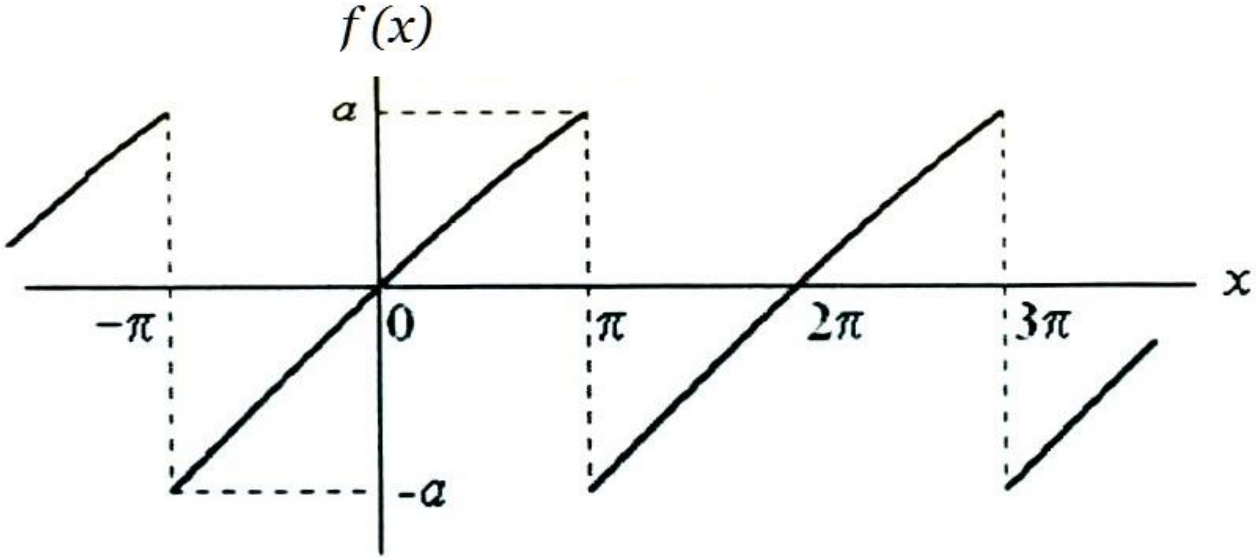


Fourier series:

$$f(x) = \frac{a}{2} - \frac{4a}{\pi^2}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{\pi} & for -\pi < x < \pi \end{cases}$$

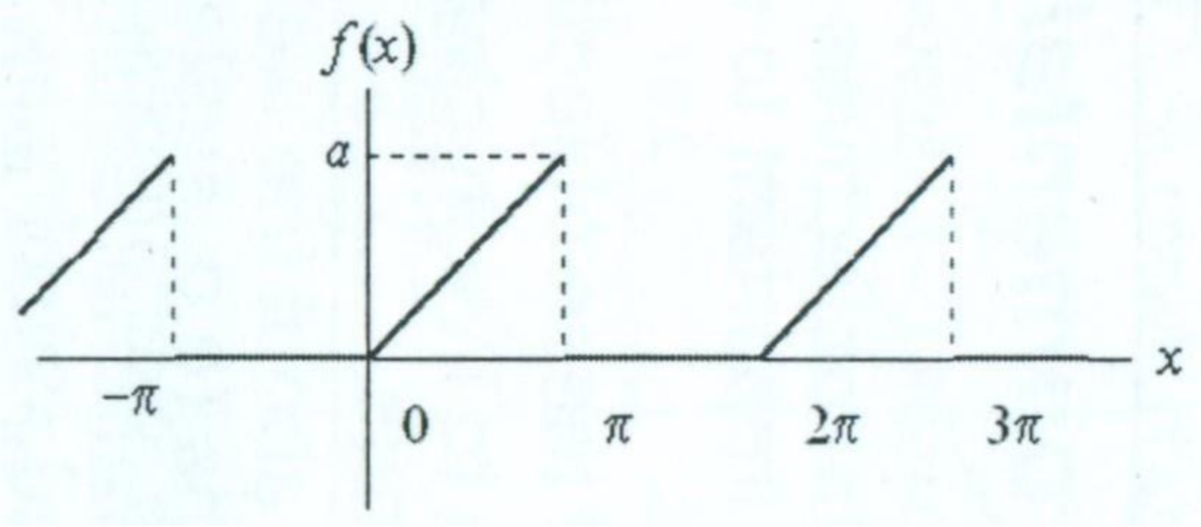


Fourier series:

$$f(x) = \frac{2a}{\pi}\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{\pi} & for\ 0 \le x \le \pi \\ 0 & for\ \pi \le x \le 2\pi \end{cases}$$

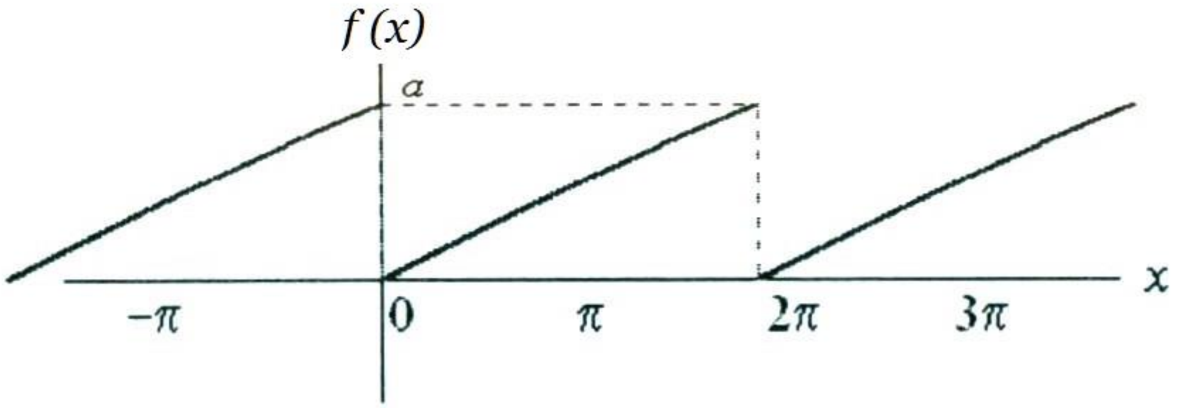


Fourier series:

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \frac{a}{\pi}\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

Function:

$$f(x) = \begin{cases} \frac{ax}{2\pi} & for\ 0 < x < 2\pi \end{cases}$$

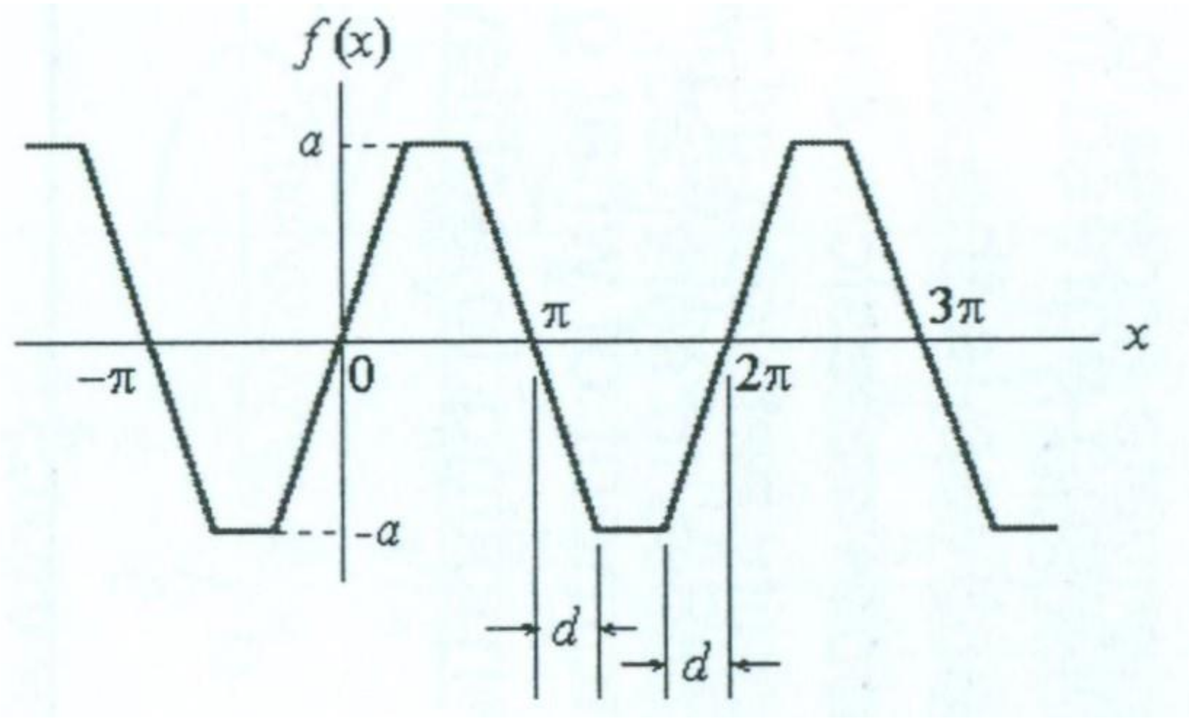


Fourier series:

$$f(x) = \frac{a}{2} - \frac{a}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$$

Function:

$$f(x) = \begin{cases} \dfrac{ax}{d} & -b \le x \le b \\ a & b \le x \le \pi - b \\ \dfrac{a(\pi - x)}{d} & for\ \pi - b < x \le \pi + b \\ -a & for\ \pi + b < x \le 2\pi - b \end{cases}$$



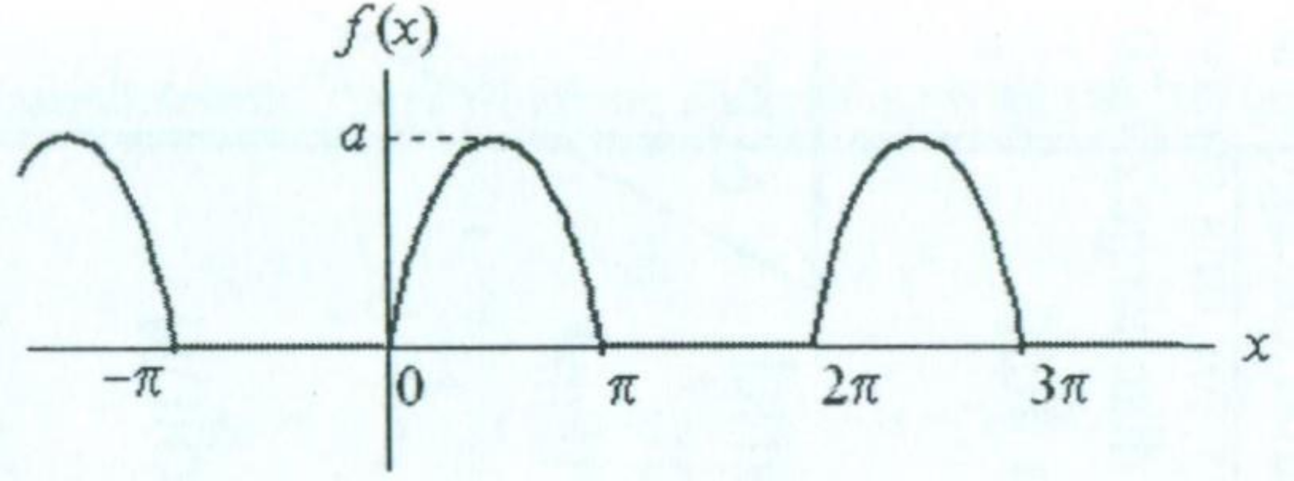
Fourier series:

$$f(x) = ?$$

Find yourself

Function:

$$f(x) = \begin{cases} a \sin x & for\ 0 \le x \le \pi \\ 0 & for\ \pi \le x \le 2\pi \end{cases}$$

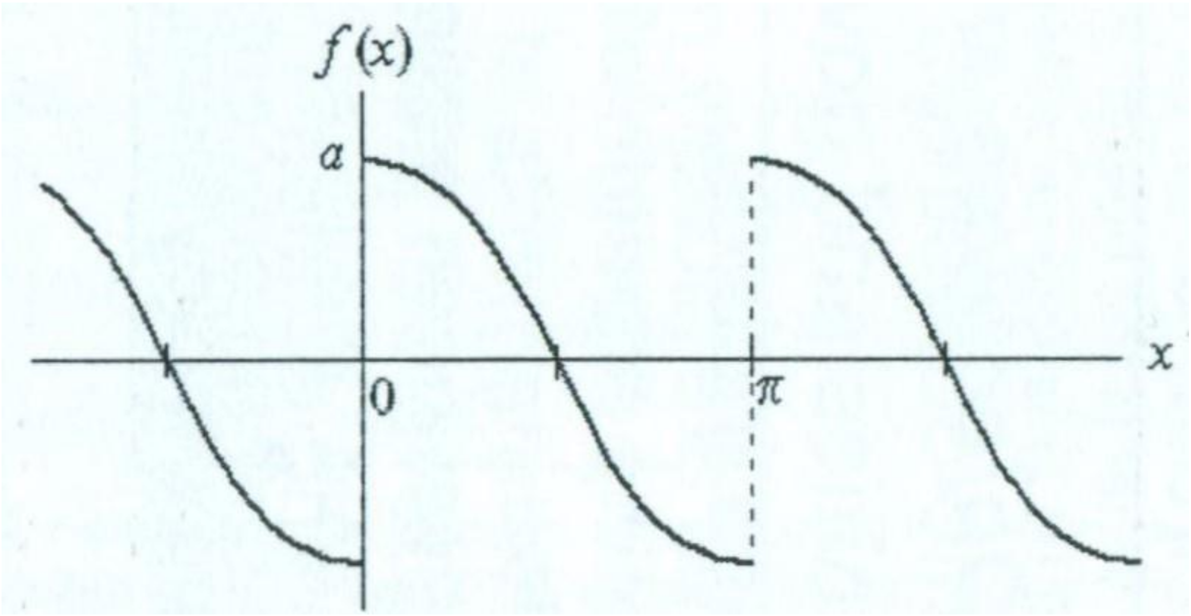


Fourier series:

$$f(x) = \frac{2a}{\pi}\left(\frac{1}{2} + \frac{\pi \sin x}{4} - \frac{\cos 2x}{1 \times 3} - \frac{\cos 4x}{3 \times 5} - \frac{\cos 6x}{5 \times 7} - \cdots\right)$$

Function:

$$f(x) = \begin{cases} a\cos x & for\ 0 < x < \pi \\ -a\cos x & for - \pi < x < 0 \end{cases}$$

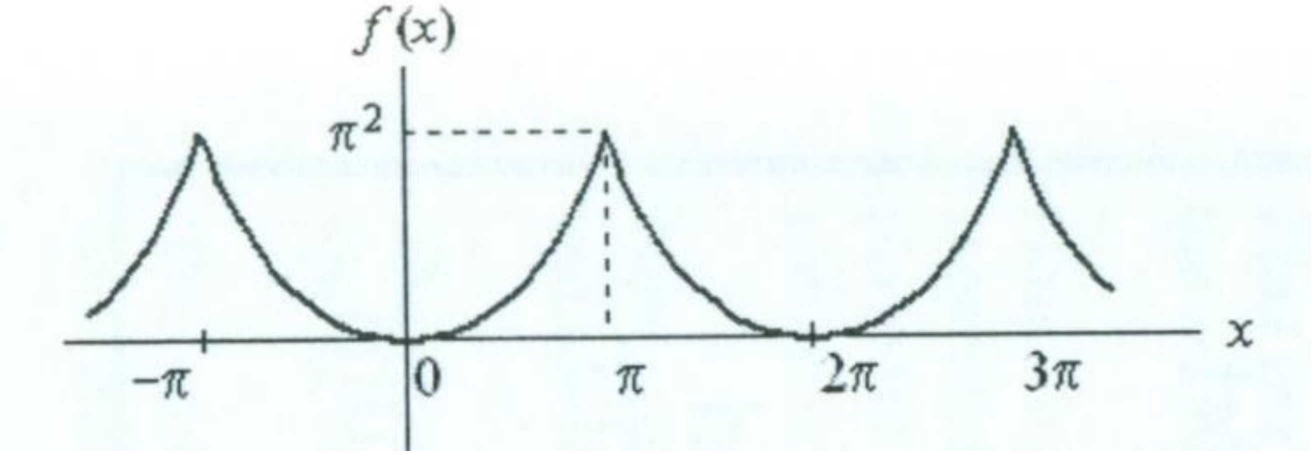


Fourier series:

$$f(x) = \frac{8a}{\pi}\left(\frac{\sin 2x}{1 \times 3} + \frac{2\sin 4x}{3 \times 5} + \frac{3\sin 6x}{5 \times 7} + \cdots\right)$$

Function:

$$f(x) = \{x^2 \quad for\ -\pi \le x \le \pi$$

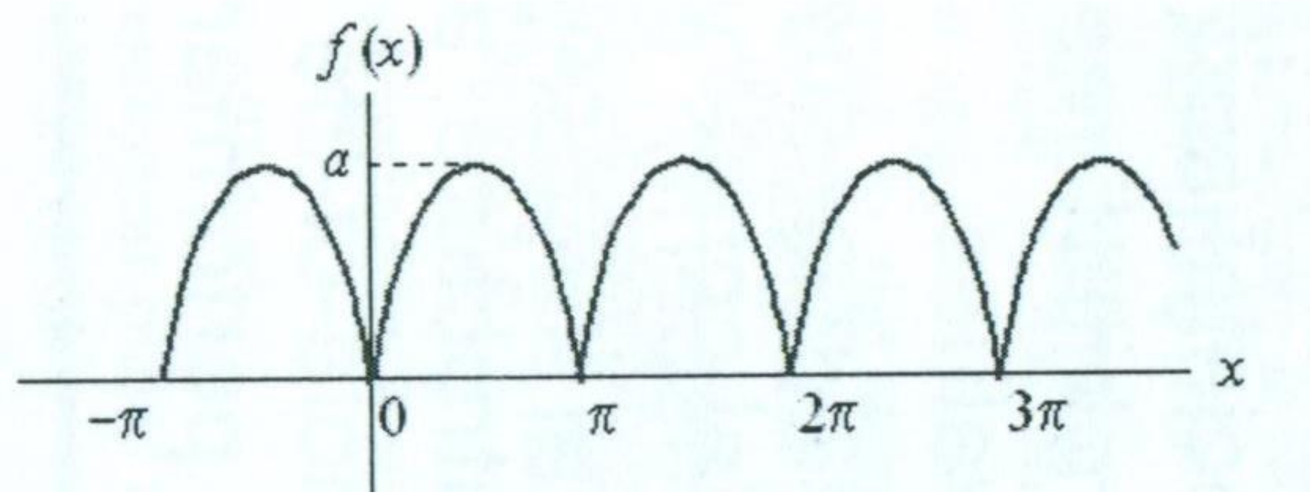


Fourier series:

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

Function:

$$f(x) = \{a|\sin x| \quad for\ -\pi < x < \pi$$



Fourier series:

$$f(x) = \frac{2a}{\pi} - \frac{4a}{\pi}\left(\frac{\cos 2x}{1\times 3} + \frac{\cos 4x}{3\times 5} + \frac{\cos 6x}{5\times 7} + \cdots\right)$$

Thank you