CE 809 - Structural Dynamics

Lecture 3: Response of SDF Systems to Harmonic Loading

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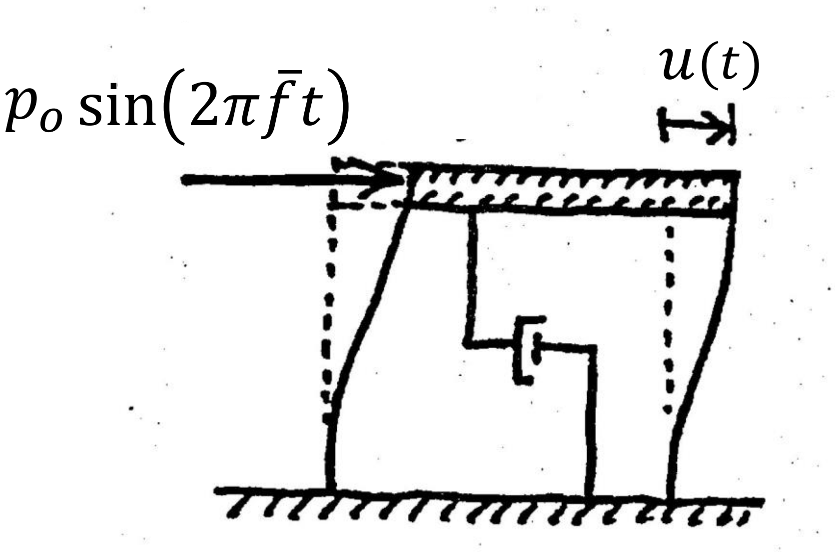
Harmonic Force

A simple structure subjected to a harmonic loading, $p(t) = p_o \sin(\bar{\omega}t) = p_o \sin\left(2\pi\bar{f}t\right)$



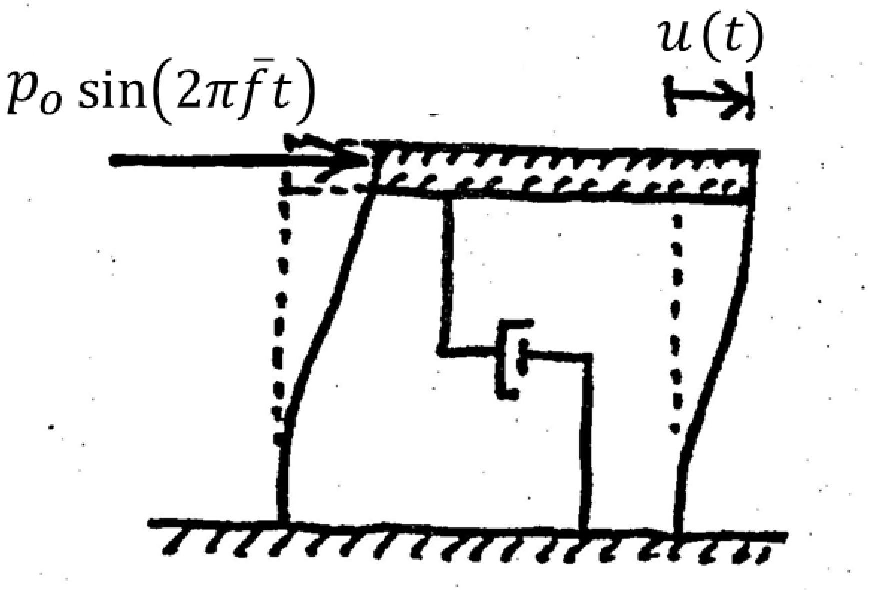
Response to Harmonic Force

In mathematics, the response is the solution of the following linear non-homogeneous differential equation:

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) \;= p_o\,\sin(\overline{\omega}t)$$

.... (1)

The solution must also satisfy the prescribed initial conditions: $u(0)$ and $\dot{u}(0)$.



A Quick Review of Basic Mathematical Concepts

Solution form:

A general solution $u(t)$ of linear nonhomogeneous differential equation is the sum of a general solution $u_h(t)$ of the corresponding homogenous differential equation and a particular solution $u_p(t)$.

$$\boxed{u(t) = u_h(t) + u_p(t)} \qquad \text{.......} (2)$$

where

$$m\,\ddot{u}_h(t) + c\,\dot{u}_h(t) + k\,u_h(t) = 0 \qquad \text{.......} (3)$$

and

$$m\,\ddot{u}_p(t) + c\,\dot{u}_p(t) + k\,u_p(t) = p_o \sin(\overline{\omega} t) \qquad \text{......} (4)$$

A Quick Review of Basic Mathematical Concepts

$u_p(t)$ is the specific response generated by the form of external force function (in this case external force function is harmonic and $u_p(t)$ is also harmonic) and $u_p(t)$ does not need to satisfy the initial conditions.

Introducing the general solution into the governing equation of motion, we obtain

$$m\left(u_h(t) \ddot{+} u_p(t)\right) + c\left(u_h(t) \dot{+} u_p(t)\right) + k\left(u_h(t) + u_p(t)\right)$$

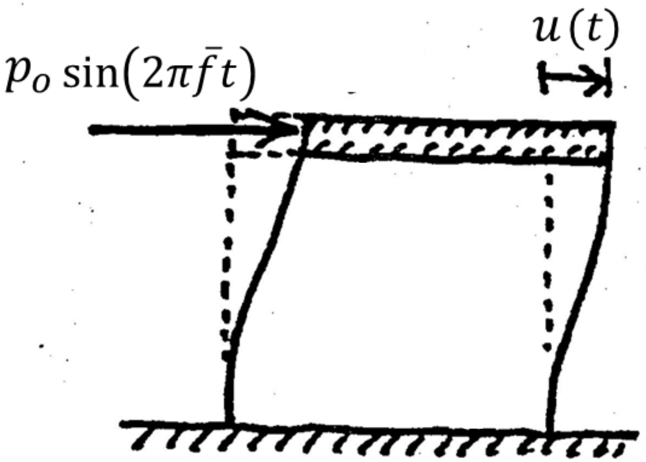
$$= \left(m\,\ddot{u}_h(t) + c\,\dot{u}_h(t) + k\,u_h(t)\right) + \left(m\,\ddot{u}_p(t) + c\,\dot{u}_p(t) + k\,u_p(t)\right)$$

$$= 0 + p_o \sin(\varpi t)$$

Response of Undamped SDF Systems

Response to Harmonic Loading

(Undamped Systems, $c = 0$)



$$m\,\ddot{u}(t) + k\,u(t) = p_o \sin(\overline{\omega} t)$$

Homogeneous (or Complementary) Solution (Undamped Systems)

From the previous section, we have already obtained $u_h(t)$ as

$$u_h(t) = A\cos(\omega t) + B\sin(\omega t)$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

Particular Solution (Undamped Systems)

The particular solution of a linear second-order differential equation governing the response of an undamped SDF system subjected to harmonic force, is of the form

$$u_p(t) = G \sin(\overline{\omega} t)$$

$$\ddot{u}_p(t) = -G\ \omega^2 \sin(\overline{\omega} t)$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

Substituting these two values of $u_p(t)$ and $\ddot{u}_p(t)$ into the governing equation of motion, we get

$$m\,\ddot{u}_p(t) + k\,u_p(t) = p_o\,\sin(\bar{\omega}t) \qquad \text{……………. (5)}$$

$$-m\,G\,\omega^2\sin(\bar{\omega}t) + k\,G\,\sin(\bar{\omega}t) = p_o\,\sin(\bar{\omega}t) \qquad \text{……………. (6)}$$

Response to Harmonic Loading

(Undamped Systems, $c \ = \ 0$)

Solving for G, we get

$$G = \frac{p_0}{k} \frac{1}{1 - (\bar{\omega}/\omega)^2} \qquad \text{……………. (7)}$$

Therefore,

$$u_p(t) = \frac{p_0}{k} \frac{1}{1 - (\bar{\omega}/\omega)^2} \sin(\bar{\omega} t) \qquad \text{……………. (8)}$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

The general solution becomes,

$$u(t) = u_h(t) + u_p(t) \qquad \text{……………… (9)}$$

$$u(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{p_0}{k}\,\frac{1}{1-(\bar{\omega}/\omega)^2}\,\sin(\bar{\omega}t) \qquad \text{……………. (10)}$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

Now we have to determine A and B,

$$\dot{u}(t) = -\omega\, A \sin(\omega t) + \omega\, B\cos(\omega t) + \frac{p_0}{k}\,\frac{\overline{\omega}}{1-(\overline{\omega}/\omega)^2}\cos(\overline{\omega}t) \qquad \text{……….} \ (11)$$

This yields,

$$u(0) = A$$

$$\dot{u}(0) = \omega\, B + \frac{p_0}{k}\,\frac{\overline{\omega}}{1-(\overline{\omega}/\omega)^2} \qquad \text{……….} \ (12)$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

Therefore,

$$\left.\begin{aligned} A &= u(0) \\ \\ B &= \frac{\dot{u}(0)}{\omega} - \frac{p_0}{k}\left[\frac{\overline{\omega}/\omega}{1-(\overline{\omega}/\omega)^2}\right] \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(13)}$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

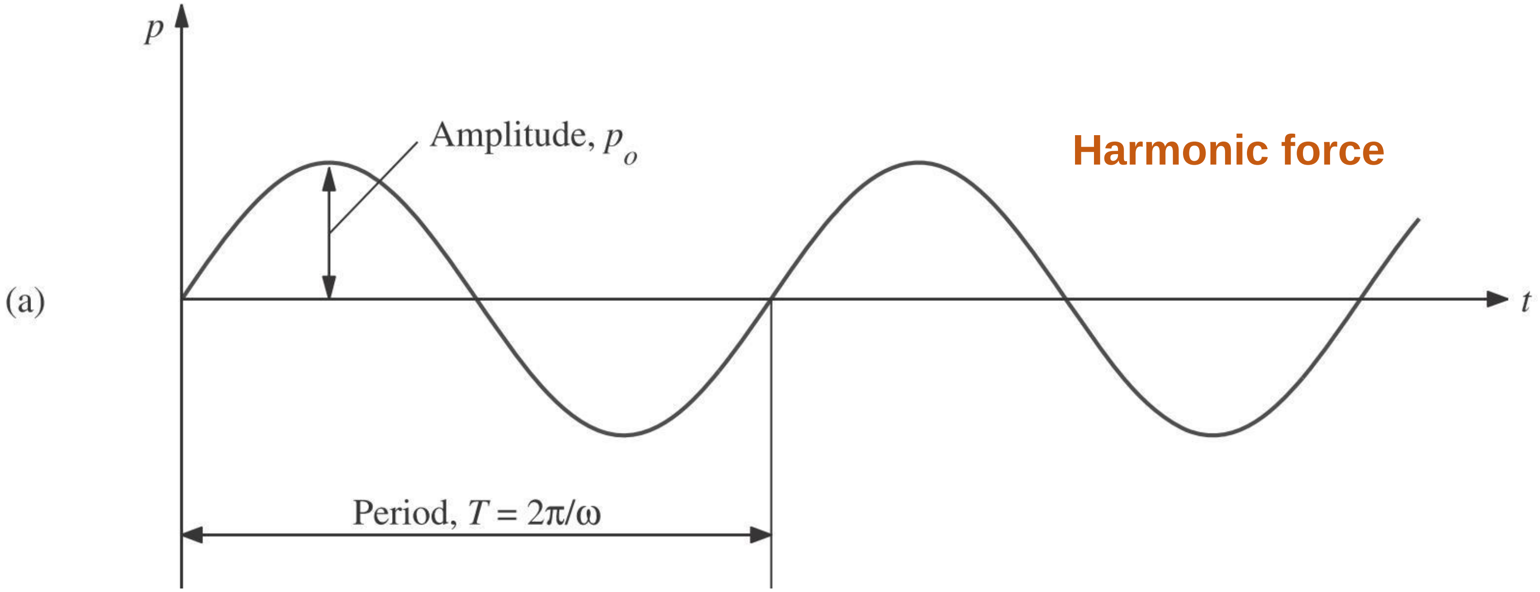
Let’s introduce $\beta = \bar{\omega}/\omega$ (frequency ratio).

The general solution becomes,

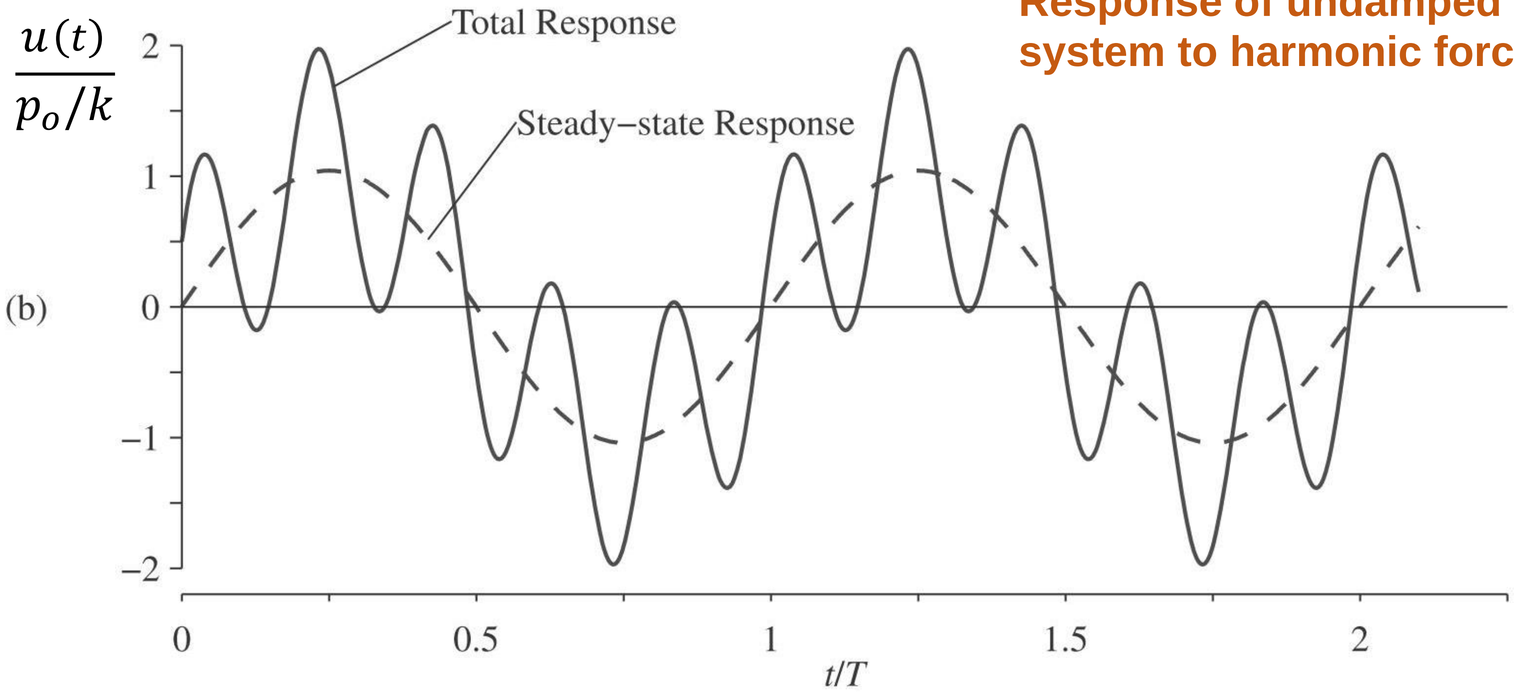
$$\boxed{u(t) = \underbrace{u(0)\cos(\omega t) + \left(\frac{\dot{u}(0)}{\omega} - \frac{p_0}{k}\frac{\beta}{1-\beta^2}\right)\sin(\omega t)}_{\textbf{Transient Vibrations}} + \underbrace{\frac{p_0}{k}\frac{1}{1-\beta^2}\sin(\bar{\omega} t)}_{\textbf{Steady-state Response}}} \quad \text{....... (14)}$$

Response to Harmonic Loading

(Undamped Systems, $c = 0$)



Harmonic force



Response of undamped system to harmonic force

$$\frac{\overline{\omega}}{\omega} = 0.2$$

$$u(0) = \frac{0.5\, p_o}{k}$$

$$\dot{u}(0) = \frac{\omega\, p_o}{k}$$

(Chopra (2012) Dynamics of Structures, 4th Edition)

Response to Harmonic Loading

(Undamped Systems, $c = 0$)

- In equation (14), $u(t)$ contains two distinct vibration components:

  (1) the $\sin(\overline{\omega}t)$ term, giving an oscillation at the forcing or exciting frequency; and

  (2) the $\sin(\omega t)$ and $\cos(\omega t)$ terms, giving an oscillation at the natural frequency of the system.

- The first of these is the **forced vibration or steady-state vibration**, for it is present because of the applied force no matter what the initial conditions.

- The latter is the **free vibration or transient vibration**, which depends on the initial displacement and velocity. It exists even if $u(0) = \dot{u}(0) = 0$.

Response to Harmonic Loading

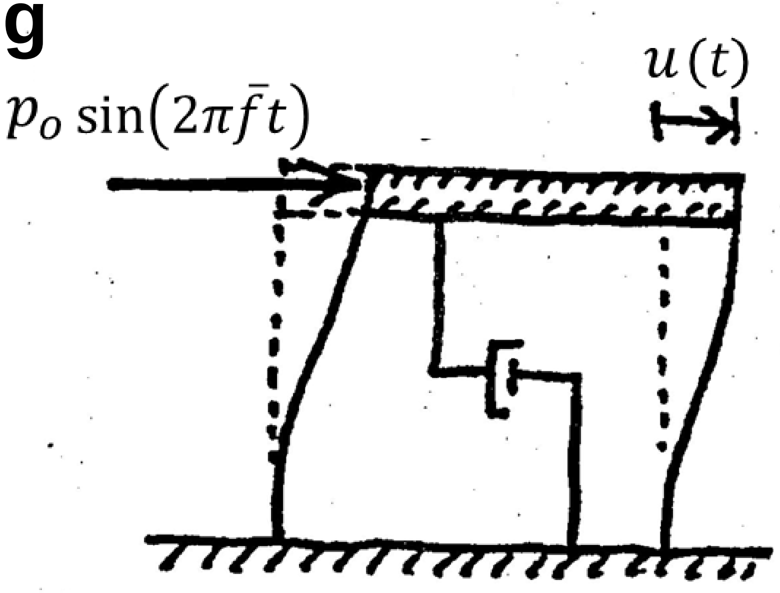
(Undamped Systems, $c = 0$)

- For the case $u(0) = \dot{u}(0) = 0$, the Equation (14) specializes to

$$u(t) = \frac{p_0}{k} \frac{1}{1-\beta^2} \left(\sin(\overline{\omega} t) - \beta \sin(\omega t)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Response of Damped SDF Systems

Response to Harmonic Loading



$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = p_o\,\sin(\bar{\omega} t) \qquad \text{................ (16)}$$

Homogeneous (or Complementary) Solution:

From the previous section, we have already obtained $u_h(t)$ as

$$\boxed{u_h(t) = e^{-\xi\,\omega\,t}\,(A\cos(\omega_D t) + B\sin(\omega_D t))} \qquad \text{................ (17)}$$

Where A and B are arbitrary constants.

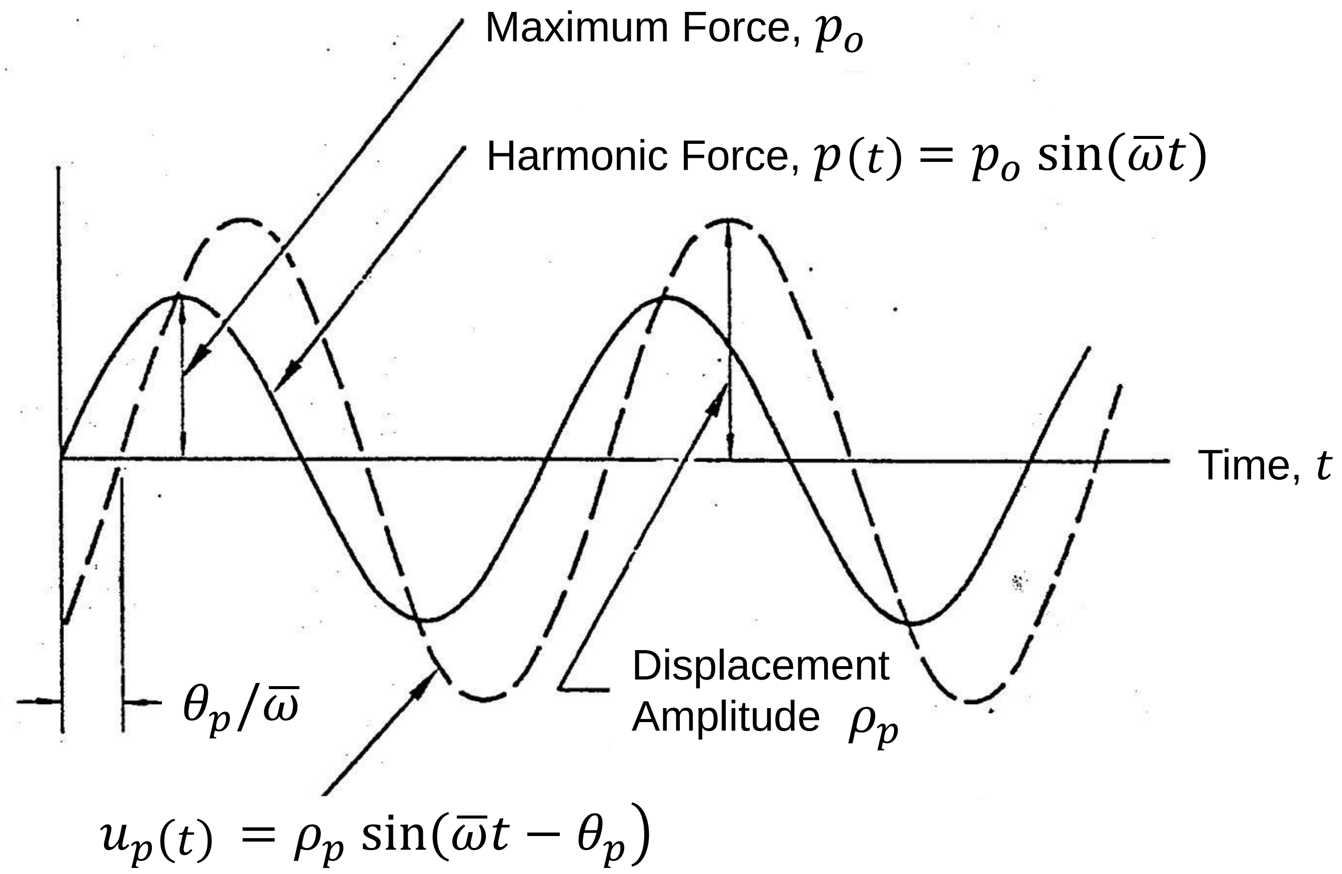
An alternate form is,

$$\boxed{u_h(t) = e^{-\xi\,\omega\,t}\,\rho_h\cos(\omega_D t - \theta_h)} \qquad \text{................ (19)}$$

Where ρ_h and θ_h are arbitrary constants

Response to Harmonic Loading

Particular solution $u_p(t)$ is the specific response generated by the form of external force function. Its form depends upon the form of dynamic loading. The specific response to a harmonic force is a harmonic function with a phase lag.



Maximum Force, p_o

Harmonic Force, $p(t) = p_o \sin(\overline{\omega} t)$

Time, t

$\theta_p / \overline{\omega}$

Displacement Amplitude ρ_p

$$u_p(t) = \rho_p \sin(\overline{\omega} t - \theta_p)$$

The particular solution for harmonic force is also harmonic with a phase lag

Response to Harmonic Loading

$$\boxed{u_p(t) = \rho_p \sin(\bar{\omega}t - \theta_p)}$$

in which ρ_p is amplitude and θ_p is phase lag.

The particular solution can also be transformed into

$$\boxed{u_p(t) = G'_1 \sin(\bar{\omega}t) + G'_2 \cos(\bar{\omega}t)} \qquad \text{................ (21)}$$

Where G'_1 and G'_2 are constants to be evaluated.

Employing the previous notations, $\omega^2 = \frac{k}{m}$ and $\xi = \frac{c}{c_c} = \frac{c}{2\,m\,\omega}$, we get

$$m\,\ddot{u}_p(t) + 2\,\xi\,m\,\omega\,\dot{u}_p(t) + m\,\omega^2 u_p(t) = p_o \sin(\bar{\omega}t) \qquad \text{................ (22)}$$

Response to Harmonic Loading

Substituting the general solution of $u_p(t)$ in equation (21) to the governing equation (22), where

$$u_{p(t)} = G'_1 \sin(\overline{\omega} t) + G'_2 \cos(\overline{\omega} t)$$

$$\dot{u}_p(t) = \overline{\omega} G'_1 \cos(\overline{\omega} t) - \overline{\omega}\, G'_2 \sin(\overline{\omega} t)$$

$$\ddot{u}_p(t) = -\overline{\omega}^2\, G'_1 \sin(\overline{\omega} t) - \overline{\omega}^2\, G'_2 \cos(\overline{\omega} t)$$

and separating the multiples of $\sin(\overline{\omega} t)$ from the multiples of $\cos(\overline{\omega} t)$ leads to

$$(-G'_1\, \overline{\omega}^2 - G'_2\, \overline{\omega}\, (2\, \xi\, \omega) + \omega^2\, G'_1) \sin(\overline{\omega} t)$$

$$+(-G'_2\, \overline{\omega}^2 + G'_1\, \overline{\omega}\, (2\, \xi\, \omega) + \omega^2\, G'_2) \cos(\overline{\omega} t) = \frac{p_o}{m} \sin(\overline{\omega} t)$$

Response to Harmonic Loading

Hence,

$$\left.\begin{aligned} -G'_1\,\overline{\omega}^2 - G'_2\,\overline{\omega}\,(2\,\xi\,\omega) + \omega^2\,G'_1 &= \frac{p_o}{m} \\ -G'_2\,\overline{\omega}^2 - G'_1\,\overline{\omega}\,(2\,\xi\,\omega) + \omega^2\,G'_2 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ (23)$$

Dividing the above two equations by ω^2 and introducing $\beta = \overline{\omega}/\omega$ (frequency ratio),

$$\left.\begin{aligned} G'_1\,(1-\beta^2) - G'_2\,(2\,\xi\,\beta) &= \frac{p_o}{k} \\ G'_2\,(1-\beta^2) + G'_1\,(2\,\xi\,\beta) &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ (24)$$

Response to Harmonic Loading

These are two simultaneous algebraic equations for two unknown $({G'}_1, {G'}_2)$. There simultaneous solution yields,

$$ {G'}_1 = \frac{p_o}{k} \frac{1 - \beta^2}{(1 - \beta^2)^2 + (2\,\xi\,\beta)^2} \qquad \text{……………. (25)} $$

$$ {G'}_2 = \frac{p_o}{k} \frac{(-2\,\xi\,\beta)}{(1 - \beta^2)^2 + (2\,\xi\,\beta)^2} \qquad \text{……………. (26)} $$

Therefore, the particular solution $u_p(t)$ is obtained as

$$ u_p(t) = {G'}_1 \sin(\bar{\omega} t) + {G'}_2 \cos(\bar{\omega} t) $$

$$ u_p(t) = \frac{p_o}{k} \frac{1}{(1 - \beta^2)^2 + (2\,\xi\,\beta)^2} \left[(1 - \beta^2) \sin(\bar{\omega} t) - 2\,\xi\,\beta \cos(\bar{\omega} t)\right] $$

Response to Harmonic Loading

This particular solution can also be written as

$$\boxed{u_p(t) = \rho_p \sin(\bar{\omega} t - \theta_p)} \quad \text{................ (27)}$$

Where,

$$\left.\begin{aligned} \rho_p &= \frac{p_o}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}} \\ \theta_p &= \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) \end{aligned}\right\} \quad \text{................ (28)}$$

Response to Harmonic Loading

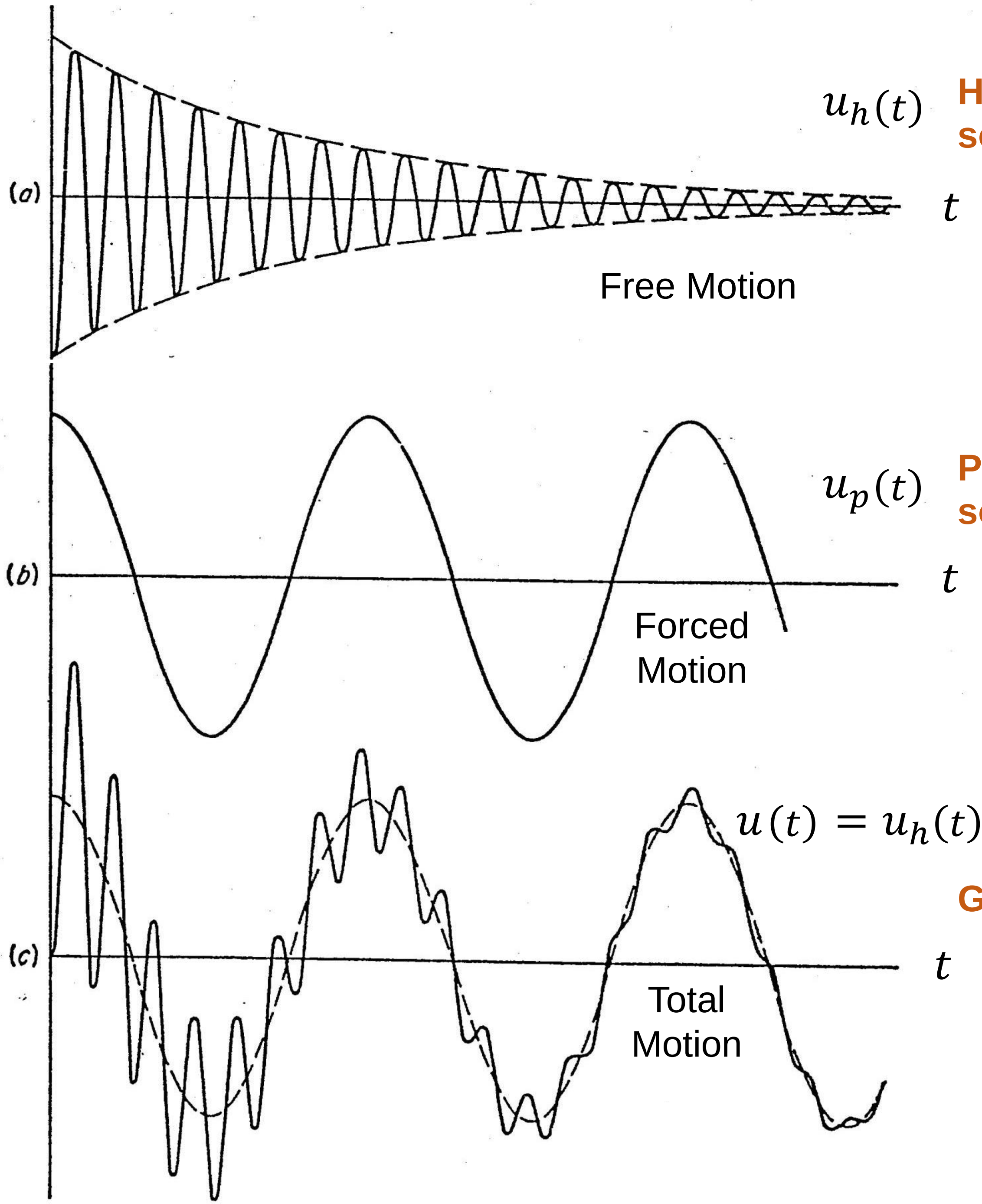
The general solution $u(t)$ is: $u(t) = u_h(t) + u_p(t)$

$$u(t) = \underbrace{e^{-\xi\,\omega\,t}\,\rho_h \cos(\omega_D t - \theta_h)} + \underbrace{\rho_p \sin(\overline{\omega} t - \theta_p)} \qquad \text{................ (30)}$$

- Free decayed oscillation at ω_D.
- ρ_h and θ_h are defined such that $u(0)$ and $\dot{u}(0)$ are satisfied.
- The oscillations of $u_h(t)$ are quickly damped and eventually become zero if the harmonic force is applied for sufficient time.

- Constant-amplitude oscillation at frequency $\overline{\omega}$ with phase θ_p different from excitation.
- This term represents the "***steady-state response***".

Response to Harmonic Loading



$u_h(t)$ **Homogeneous solution**

Free Motion

$u_p(t)$ **Particular solution**

Forced Motion

$u(t) = u_h(t) + u_p(t)$

General solution

Total Motion

Note that it this case $u(0)$ = 0 and $\dot{u}(0)$ = 0.

Response to Harmonic Loading

Steady-state Response

After sufficient time has passed, $u(t) \rightarrow u_p(t)$

Therefore, $u_p(t)$ is the “steady-state response”

$$u_p(t) = \rho_p \sin(\bar{\omega} t - \theta_p) \qquad \ldots\ldots\ldots\ldots\ldots (31)$$

Where

$$\rho_p = \frac{p_o}{k} D$$

$$D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}}$$

$$\theta_p = \tan^{-1}\left(\frac{2\,\xi\,\beta}{1-\beta^2}\right)$$

Response to Harmonic Loading

Steady-state Response

- The term p_o/k is the maximum static displacement (u_o^{st}). It is the displacement of structure that would occur if the maximum force p_o were applied as a static force.

- D is a dimensionless factor known as the "***dynamic magnification factor***" or "***dynamic response factor***".

Response to Harmonic Loading

Steady-state Response

Maximum dynamic displacement (ρ) = Maximum static displacement x Dynamic magnification factor

Dynamic magnification factor, $D = \dfrac{1}{\sqrt{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}}$

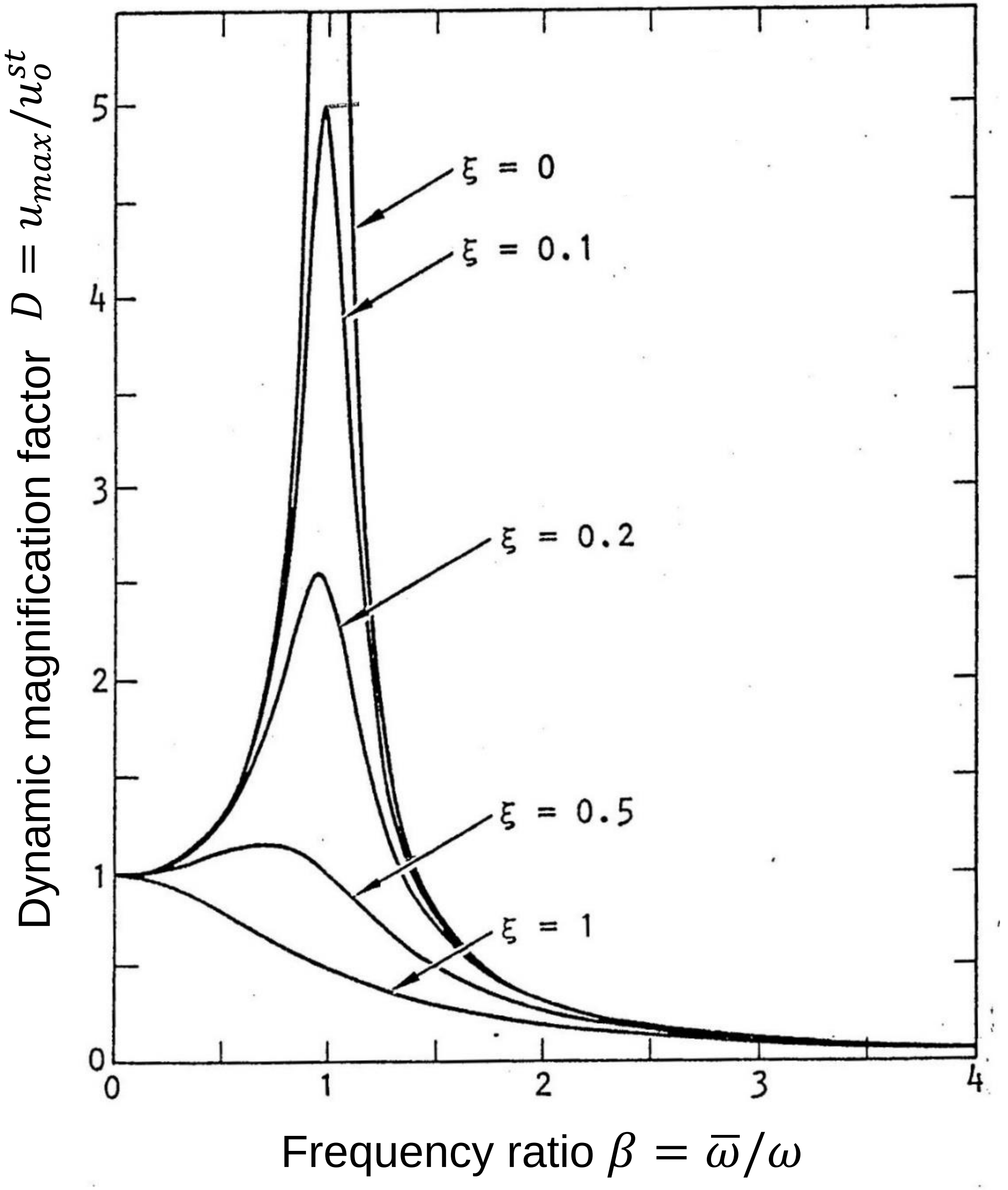
D is a function of (1) Frequency ratio $\beta = \overline{\omega}/\omega$ and (2) Critical damping ratio ξ

A plot of the amplitude of a response quantity against the excitation frequency is called a **frequency-response curve**.

Response to Harmonic Loading

Steady-state Response

- Figure shows the plot of D against β for structures with ξ = 0, 0.1, 0.2, 0.5 and 1.
- Damping reduces D, and hence the deformation amplitude at all excitation frequencies.



Response to Harmonic Loading

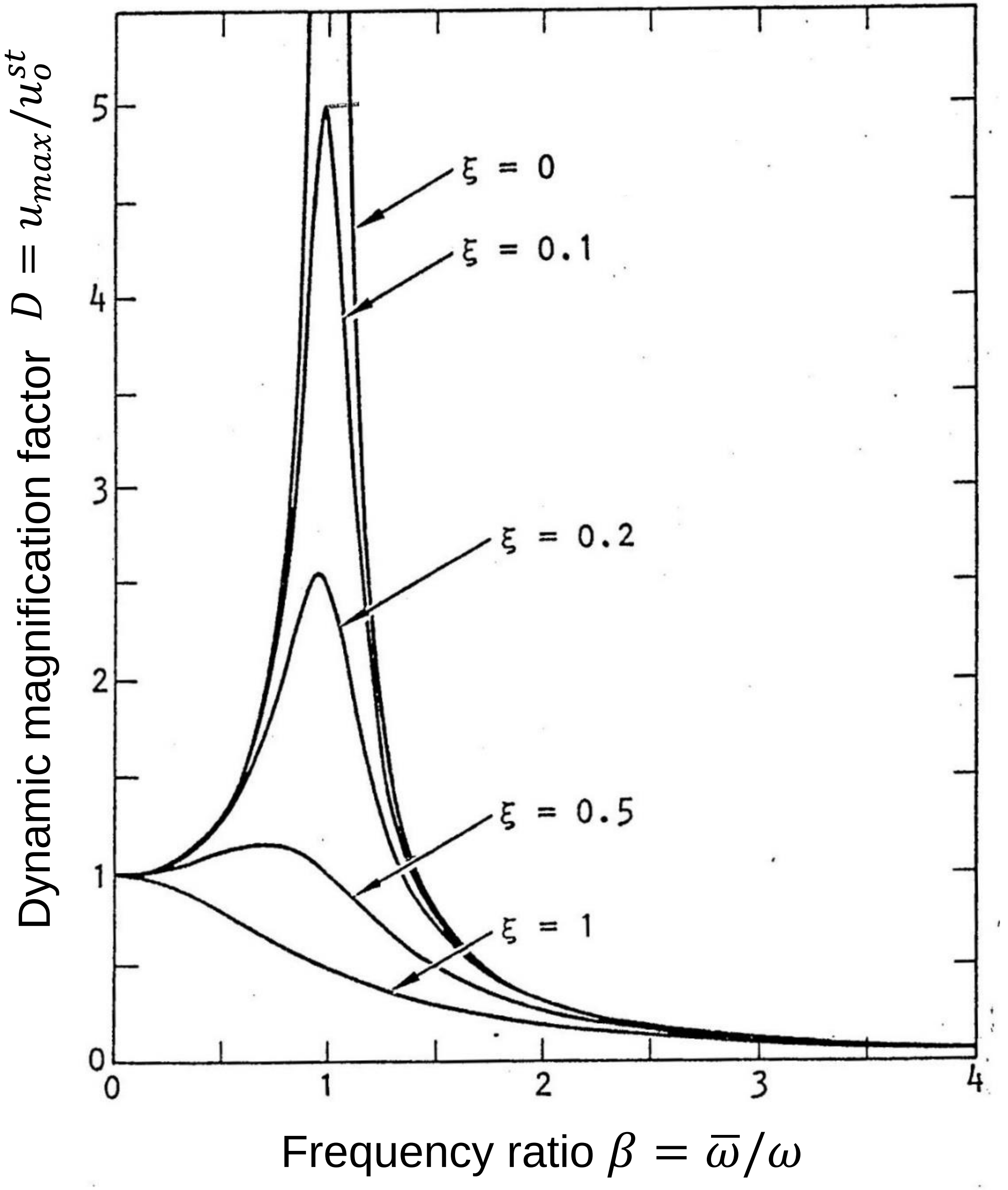
Several observations can be made:

(1) When β approaches to zero, $D \rightarrow 1$, and the dynamic displacement amplitude is about the same as the static one.

In the other words, if the forcing frequency $\overline{\omega}$ is much lower than the natural frequency of the structure ω, the dynamic effects are negligible.

The displacement is controlled by the stiffness of structure, with little effect of mass and damping, so we call this range (β → 0) as **"pseudo static"** range.

$$\rho \cong u_o^{st} = \frac{p_o}{k}$$



Response to Harmonic Loading

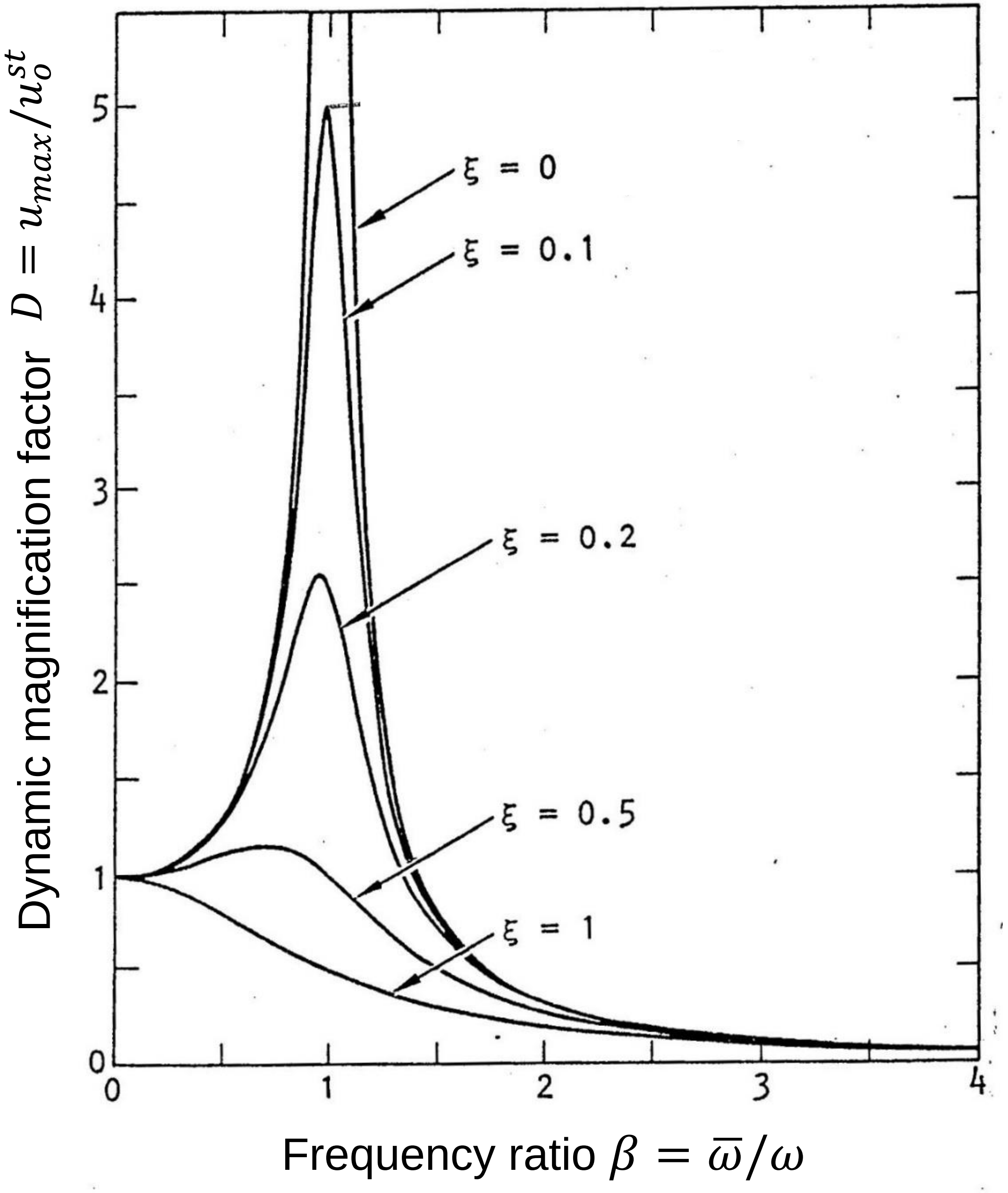
(2) On the other extreme, when β >> 1, D → 0.

If the forcing frequency $\overline{\omega}$ is much higher than the natural frequency of the structure ω, the displacement approaches to zero.

In this extreme, the inertia force dominates. So we call this range **“inertial range”**.

This result implies that the response is controlled by the mass of the system.

$$u_o \cong u_o^{st} / \beta^2$$



Response to Harmonic Loading

(3) Between the two extremes, there is a range where the displacement can be very large when damping ratio is low.

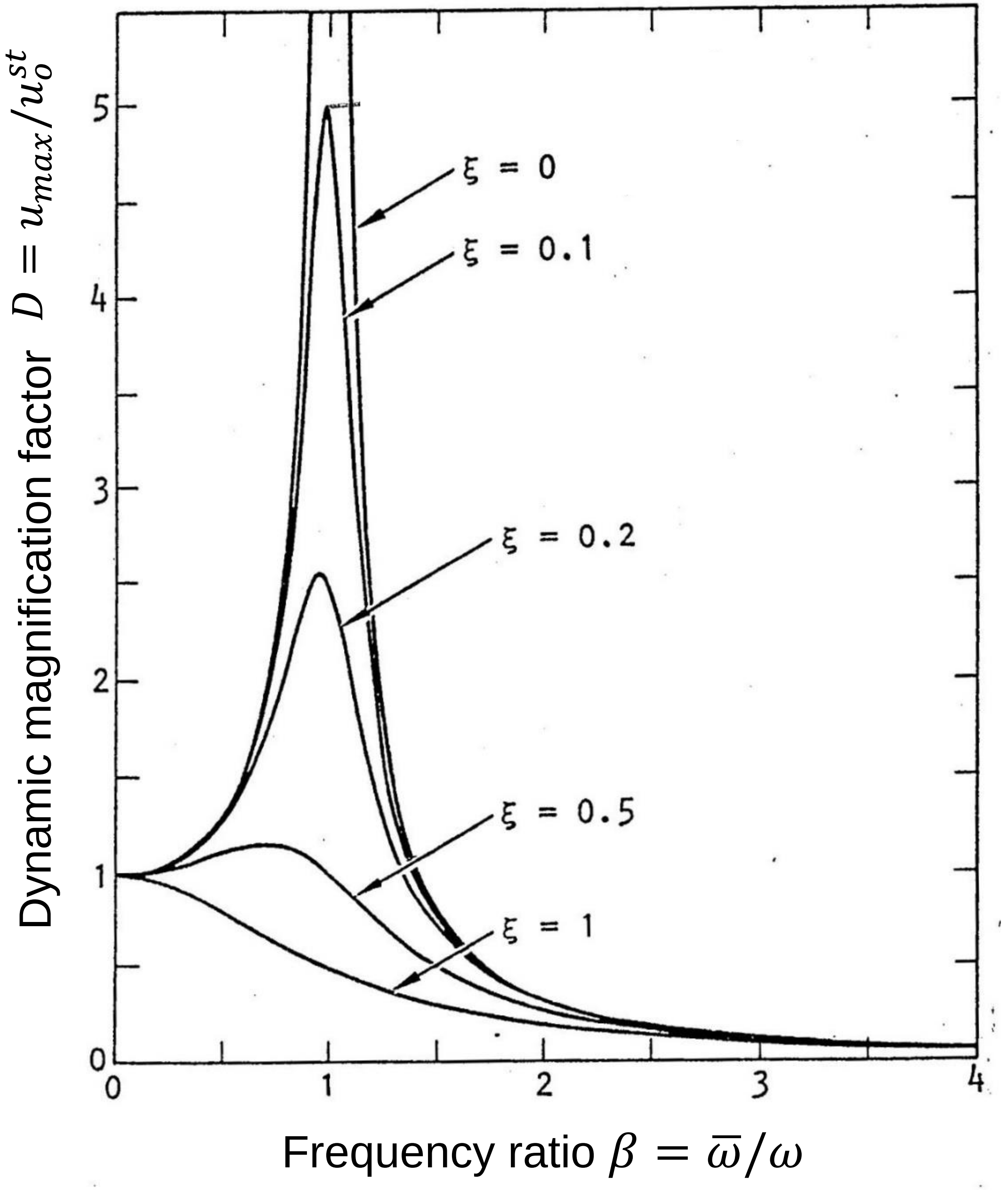
This is the range where β is close to 1.

At $\beta = 1$, D → peak, i.e. a small force can produce a very large response.

$$D = \frac{1}{2\,\xi}$$

This result implies that the response is controlled by the damping of the system. Dynamic magnification factor is inversely proportional to damping.

In this range, the damping force plays a very crucial role. So, we call this range **"resonant range"**.



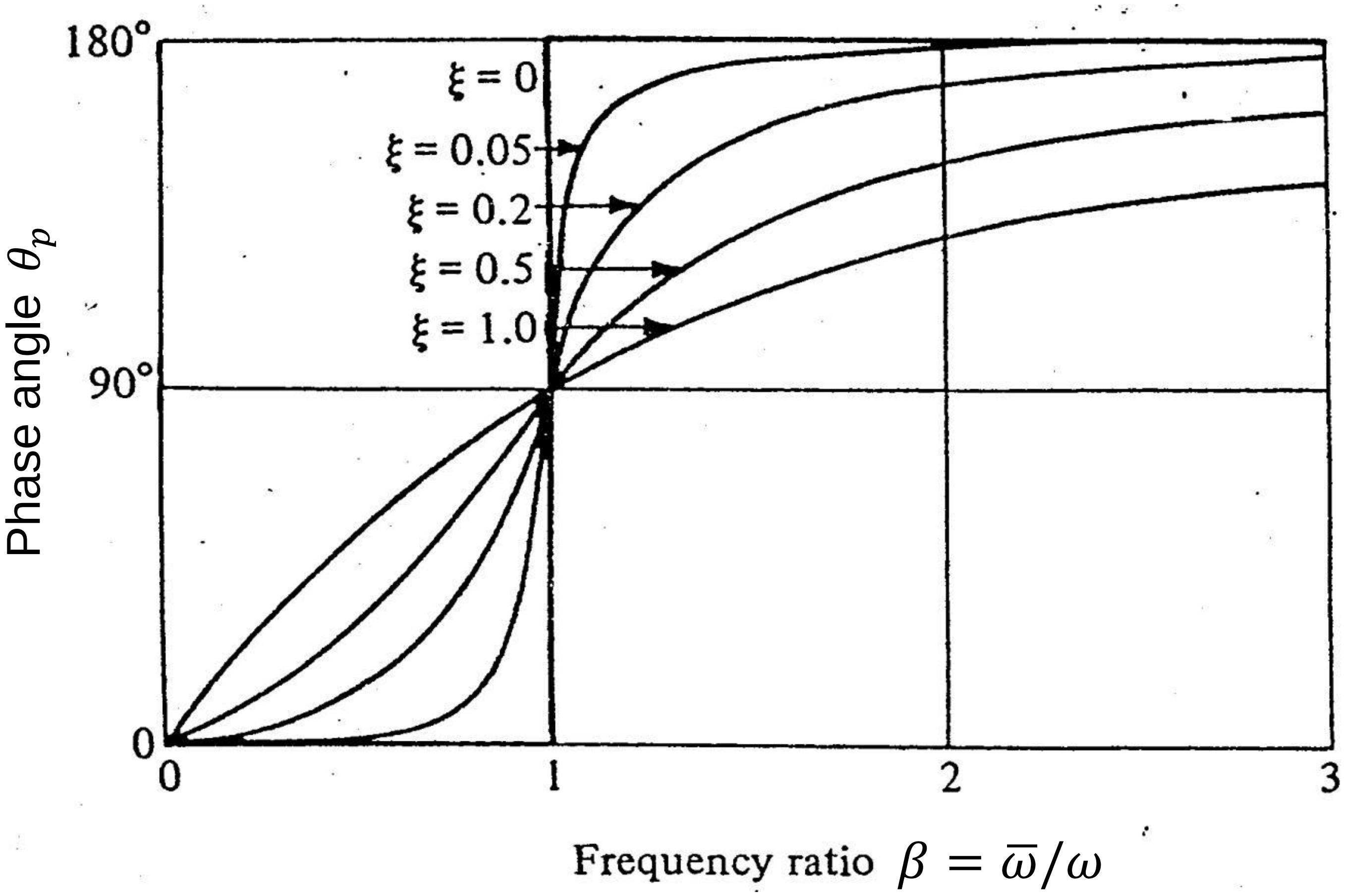
Resonant Amplification

To give you some ideas about this “resonant amplification”,

- ξ of steel structures ≈ $0.01,\ \ D = 1/(2 \times 0.01) = 50$
- ξ of concrete structures ≈ $0.05,\ \ D = 1/(2 \times 0.05) = 10$
- ξ of tall towers (300 m to 400 m high), long span bridges (300 m up span)≈ $0.005,\ D = 100$

Phase Angle (θ_p)

- The phase angle θ_p, which defines the time by which the response lags behind the force, varies with $\beta = \overline{\omega}/\omega$ as shown in Figure.
- It is examined next for the same three regions of the excitation-frequency scale:



(Chopra (2012) Dynamics of Structures, 4th Edition)

Phase Angle (θ_p)

1) If $\beta = \overline{\omega}/\omega \ll 1$ (i.e., the force is “slowly varying”), θ_p is close to 0° and the displacement is essentially in phase with the applied force. When the force acts to the right, the system would also be displaced to the right.

2) If $\beta = \overline{\omega}/\omega \gg 1$ (i.e., the force is “rapidly varying”), θ_p is close to 180° and the displacement is essentially of opposite phase relative to the applied force. When the force acts to the right, the system would be displaced to the left.

3) If $\beta = \overline{\omega}/\omega = 1$ (i.e., the forcing frequency is equal to the natural frequency), $\theta_p = 90^\circ$ for all values of ξ, and the displacement attains its peaks when the force passes through zeros.

(Chopra (2012) Dynamics of Structures, 4th Edition)

Additional explanation on *pseudo-static*, *inertial* and *resonant* ranges

Let us consider the equation of motion.

$$f_I(t) \; + f_D\,(t) + f_s\,(t) = p_o\,\sin(\omega t)$$

The left-hand side of the equation contains three structural dynamic forces. The right-hand side is an external force.

The proportions of these three forces (at steady-state condition) are derived as follows.

$$\left.\begin{aligned} u(t) &= \rho\,\sin(\overline{\omega}t-\theta) \\ \dot{u}\,(t) &= \overline{\omega}\,\rho\,\cos(\overline{\omega}t-\theta) \\ \ddot{u}\,(t) &= -\overline{\omega}^2\,\rho\,\sin(\overline{\omega}t-\theta) \end{aligned}\right\} \text{At steady-state condition}$$

$u(t)$ and $\ddot{u}(t)$ are in opposite phase.

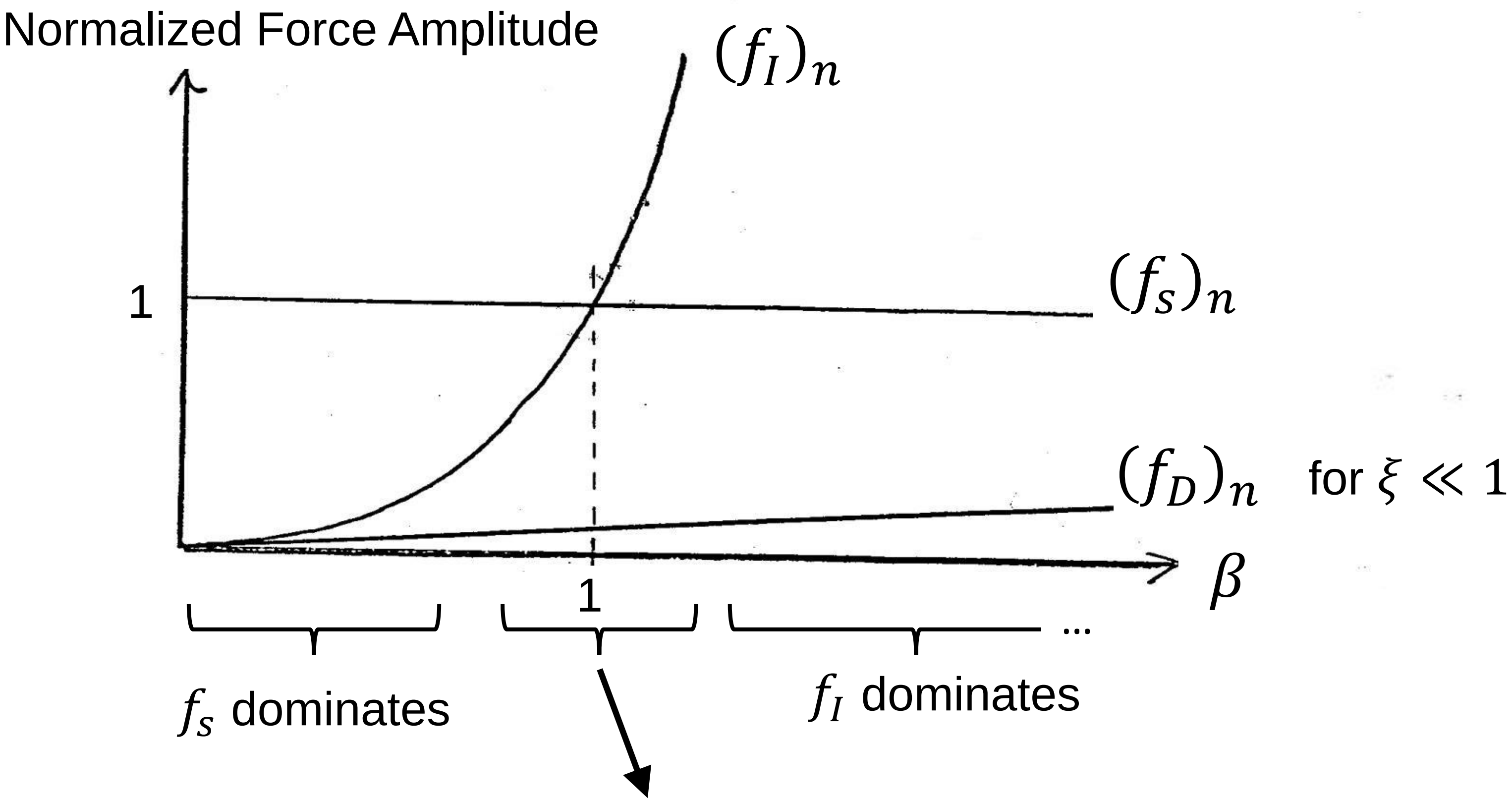
Additional explanation on *pseudo-static*, *inertial* and *resonant* ranges

$$f_s = k\,u(t) \qquad |f_s|_{\max} = k\,\rho \qquad (f_s)_n = \frac{|f_s|_{\max}}{k\,\rho} = 1$$

$$f_s = c\,\dot{u}(t) \qquad |f_D|_{\max} = 2\,\xi\,\beta\,k\,\rho \qquad (f_D)_n = \frac{|f_D|_{\max}}{k\,\rho} = 2\,\xi\,\beta$$

$$f_s = m\,\ddot{u}(t) \qquad |f_I|_{\max} = \beta^2\,k\,\rho \qquad (f_I)_n = \frac{|f_I|_{\max}}{k\,\rho} = \beta^2$$

Additional explanation on *pseudo-static*, *inertial* and *resonant* ranges



Both f_s and f_I are the major forces

But they are in opposite phase, hence cancelling each other

The remaining f_D, which is relatively weak, becomes more important.

Additional explanation on *pseudo-static*, *inertial* and *resonant* ranges

- Close to $\beta = 1$, both f_S and f_I are the major forces, but they are in opposite phase, hence cancelling each other. The remaining f_D which is relatively weak force becomes more important in this middle range.

- At $\beta = 1$, the equation of motion becomes $f_D = p_o \sin(\omega t)$. In order to satisfy this equilibrium, large ρ is developed → resonance.

$$|f_D|_{\max} = 2\,\xi\,\beta\,k\,\rho = p_o$$

- The term $2\,\xi$ is small, and hence ρ needs to be large.

Resonant Response

To gain more understanding in the nature of resonant response, let us consider the general solution $u(t)$ at $\beta = 1$:

$$u(t) = e^{-\xi\,\omega\,t}\,\rho_h\cos(\omega t - \theta_h) + \frac{p_o}{k}\left(\frac{1}{2\,\xi}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$

Note that $\sin(\omega t - \frac{\pi}{2}) = -\cos(\omega t)$

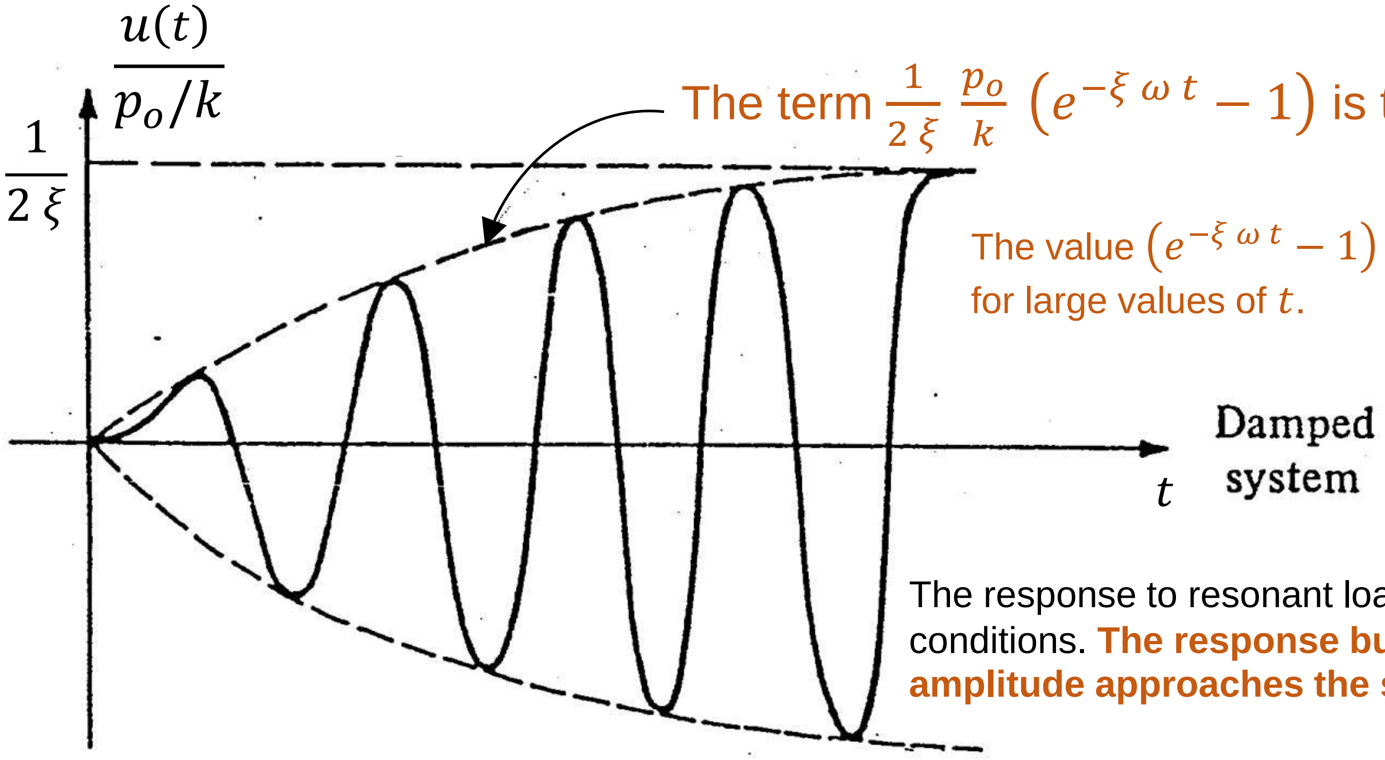
Assume that the structure initially has no motion i.e. $u(0) = 0$ and $\dot{u}(0) = 0$. With these specified initial conditions, ρ_h and θ_h can be determined and, we finally obtain

$$u(t) = \frac{1}{2\,\xi}\,\frac{p_o}{k}\left(e^{-\xi\,\omega\,t}\left[\cos(\omega_D t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_D t)\right] - \cos(\omega t)\right)$$

Resonant Response

For lightly damped systems,

$$u(t) \cong \frac{1}{2\,\xi}\,\frac{p_o}{k}\left(e^{-\xi\,\omega\,t}-1\right)\cos(\omega t)$$



The term $\frac{1}{2\,\xi}\,\frac{p_o}{k}\left(e^{-\xi\,\omega\,t}-1\right)$ is the envelope function.

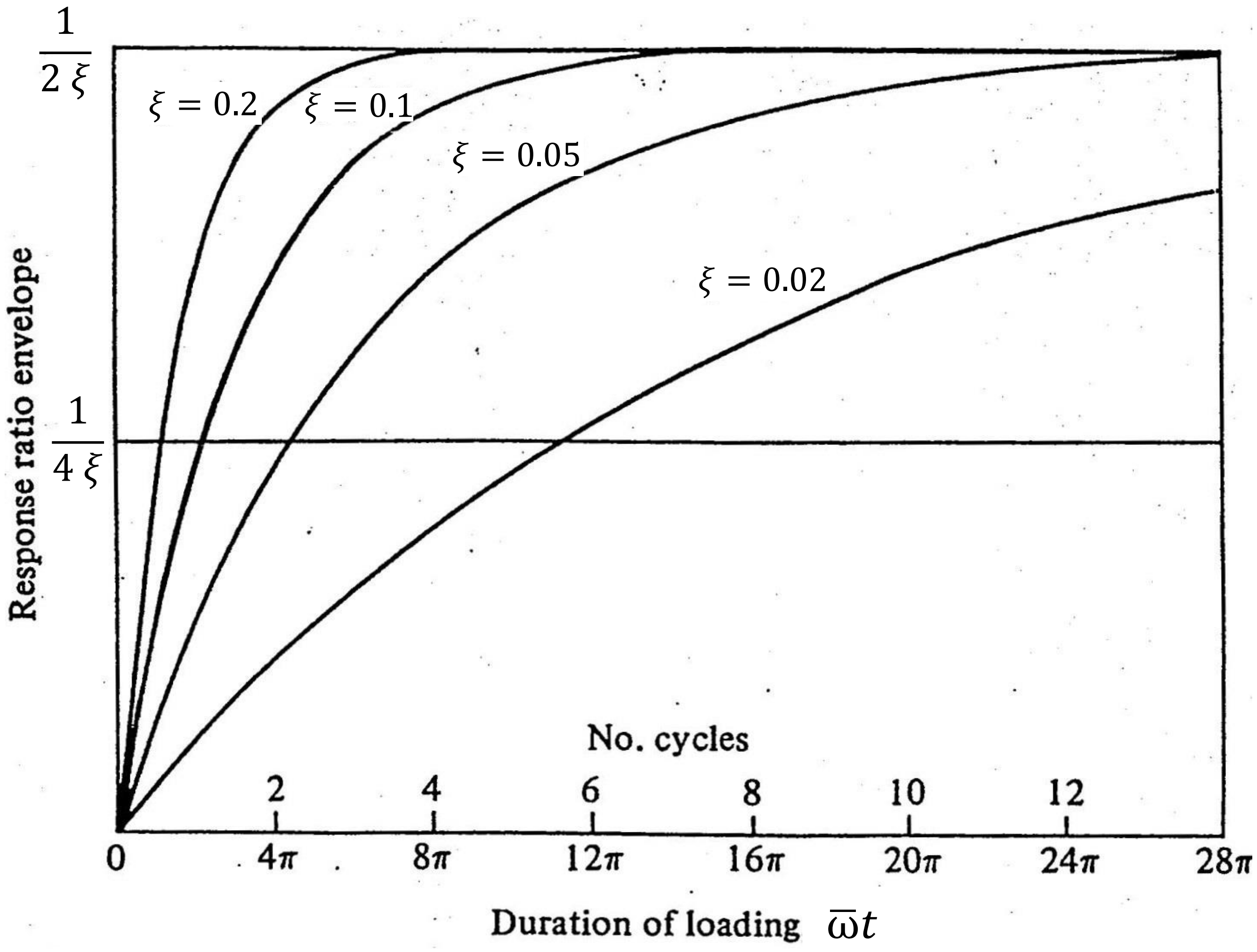
The value $\left(e^{-\xi\,\omega\,t}-1\right)$ starts from 0 and approach -1 for large values of t.

The response to resonant loading $\beta = 1$ for at-rest initial conditions. **The response builds up gradually until the amplitude approaches the steady-state value.**

Resonant Response

- For highly damped systems, it takes only a few cycles to reach the peak.

- For lowly damped systems, it may take large number of cycles to reach the peak.



The rate of buildup of resonant response from rest

Resonant Response

Therefore, in order that large resonant response to be fully developed, **three conditions** have to be met:

a. Frequency tuning, $\beta \cong 1$

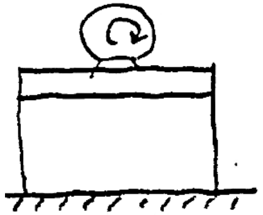
b. Low damping ratio, $\xi \ll 1$

c. Sufficiently long duration of excitation

Wind Loading

Vortex flow generates harmonic force

Machine Loading



Unbalance motor

Examples of harmonic loading which can cause resonant response

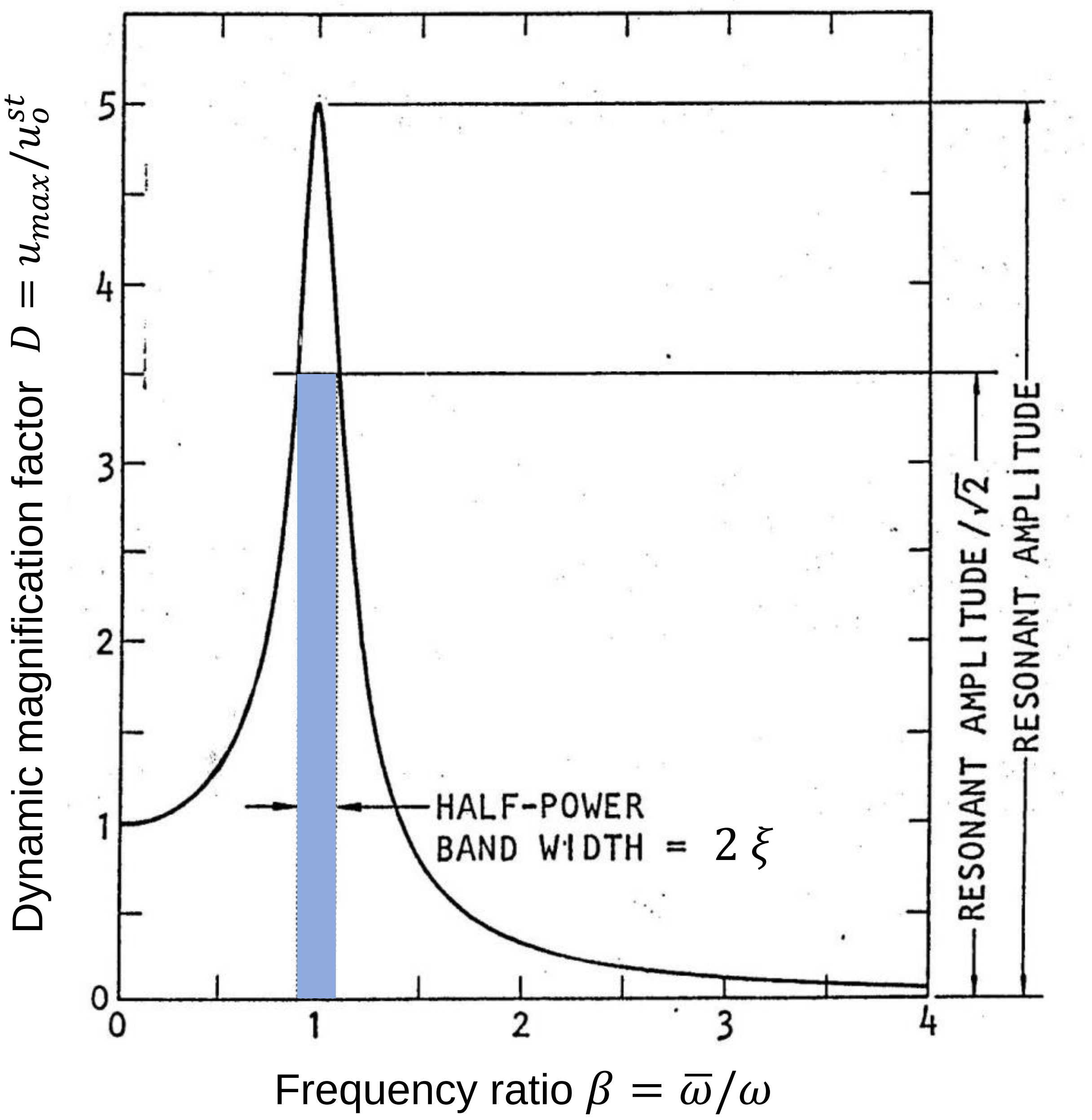
Evaluation of Damping from Resonant Curve

- “Resonance” makes ‘dynamic response’ much different from ‘static response’.
- “Resonance magnification” is governed by “damping”
- But it is usually not feasible to determine the damping “c” for a given structure. This is *a major source of error* in dynamic analysis.
- So the value of “c” is usually assumed based on past experiences.
- Damping “c” can be evaluated directly from experiment.
- One way to experimentally determine the damping “c” is “free vibration decay”.
- Another way to estimate the damping “c” is to use the “**frequency-response curve**”. Frequency-response curve is a plot of the amplitude of a response quantity against the excitation frequency.

The Evaluation of damping from force vibration tests

$$\xi = \frac{1}{2\,D}$$

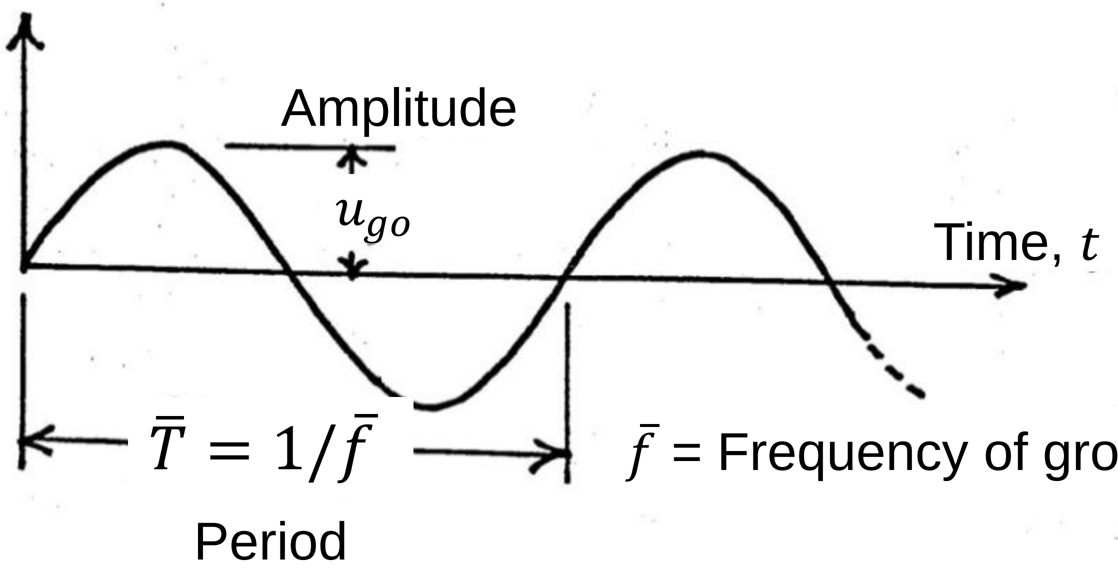
$$\xi = \frac{1}{2}\,\frac{u_o^{st}}{u_{o(\overline{\omega}=\omega)}}$$



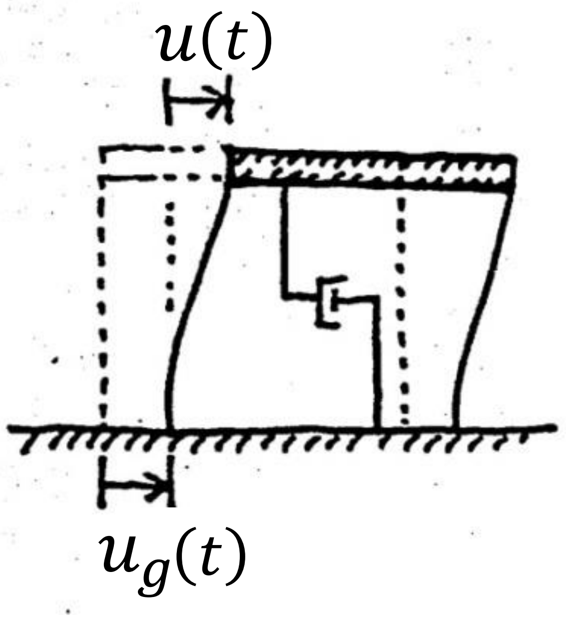
Response to Harmonic Ground Motions

Harmonic ground motion is represented by

$$u_g(t) = u_{go}\sin(2\pi\bar{f}t)$$



$\bar{f}$ = Frequency of ground motion



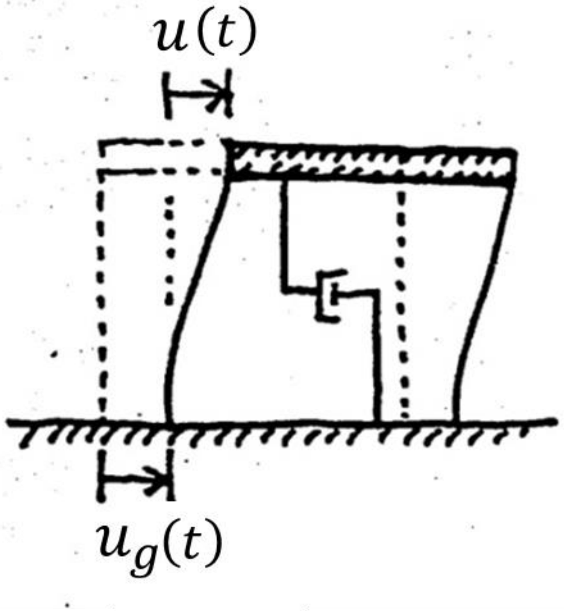
Response to Harmonic Ground Motions

Effective Force:

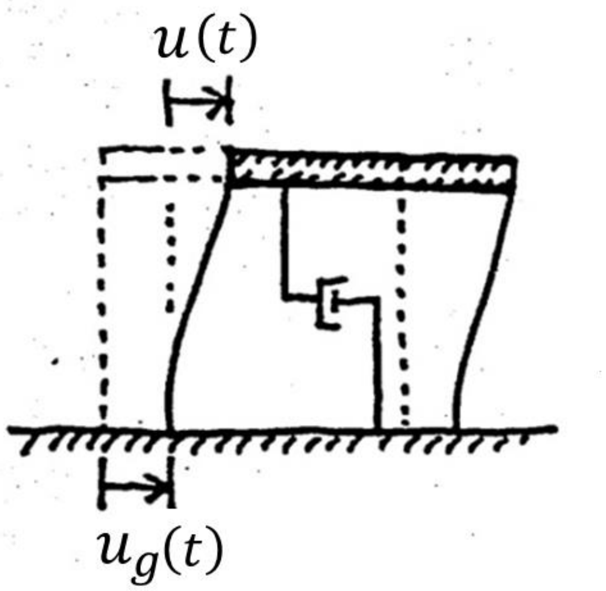
$$P_{eff}(t) = -m\frac{d^2u(t)}{dt^2} = m\left(2\,\pi\,\bar{f}\right)^2 u_{go}\sin(2\pi\bar{f}t)$$

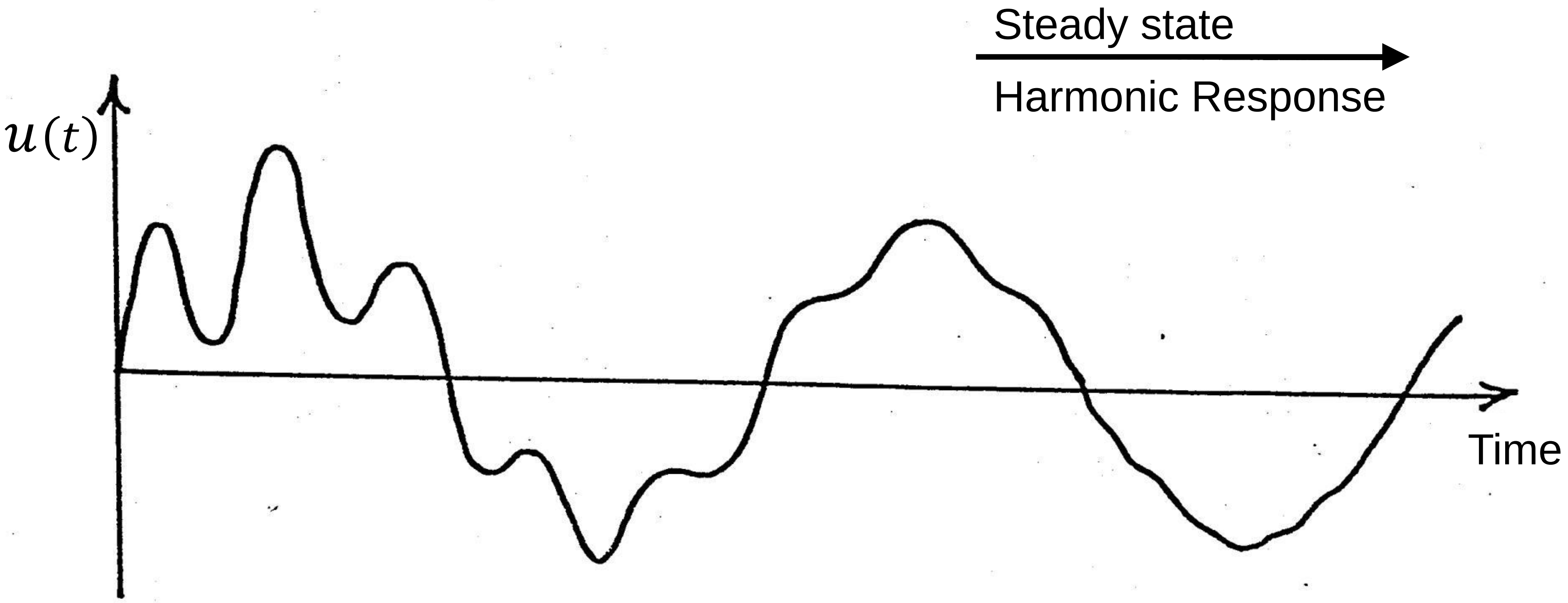
Equation of motion:

$$m\,\frac{d^2u(t)}{dt^2} + c\frac{du(t)}{dt} + k\,u(t) = P_{eff}(t) = m\left(2\,\pi\,\bar{f}\right)^2 u_{go}\sin(2\pi\bar{f}t)$$



Response to Harmonic Ground Motions





Steady state

Harmonic Response

Response to Harmonic Ground Motions

At Steady stage:

$$u(t) = R\, u_{go} \sin(2\pi\bar{f}t - \phi)$$

ϕ = phase lag

R = Dynamic amplification factor (function of frequency ratio and damping ratio)

Response to Harmonic Ground Motions

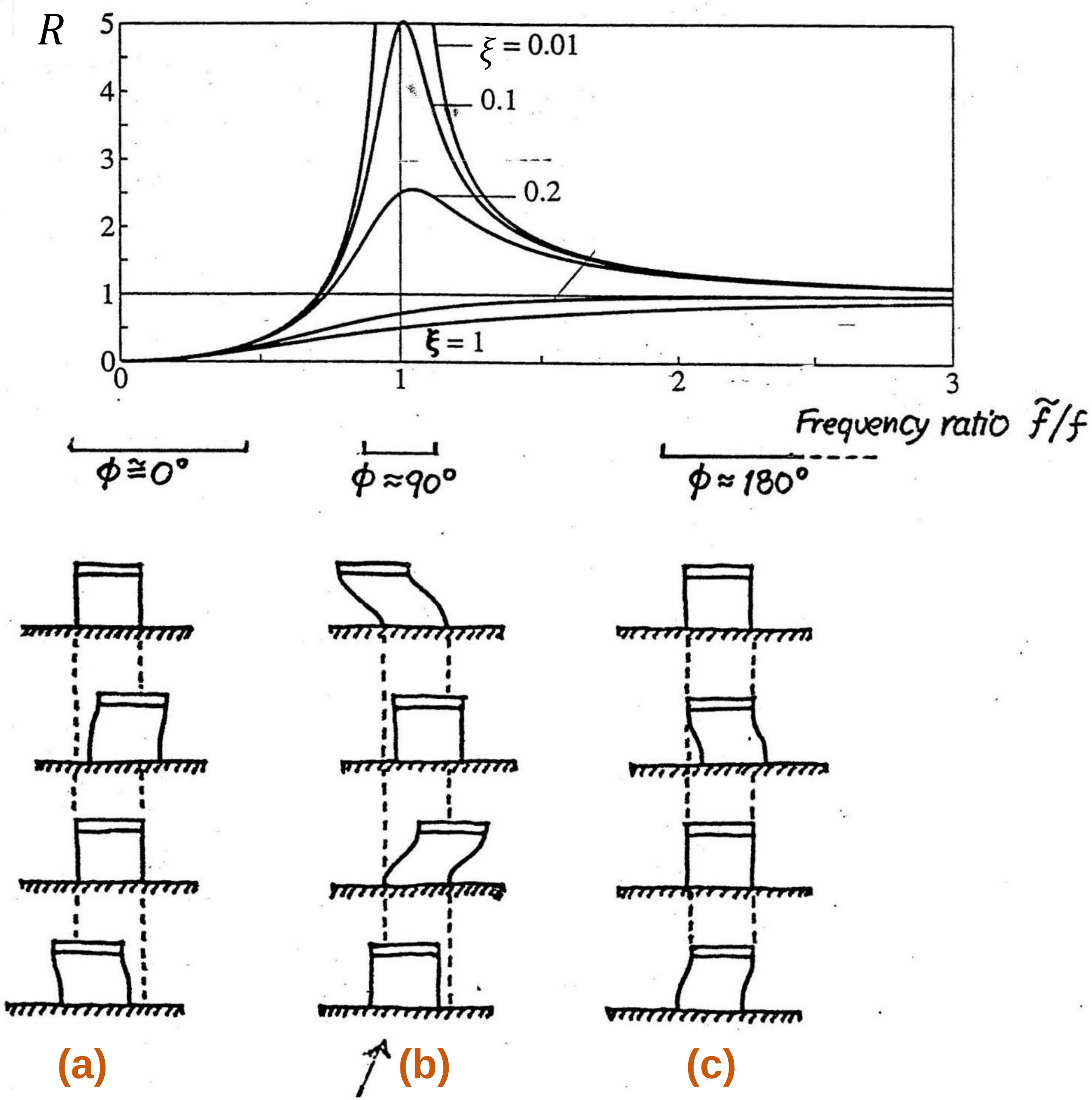
$$R = \frac{\left(\frac{\bar{f}}{f}\right)^2}{\sqrt{\left(1 - \left(\frac{\bar{f}}{f}\right)^2\right)^2 + \left(2\,\xi\,\frac{\bar{f}}{f}\right)^2}}$$

For $R = 1$, the structure will have same amplitude of shaking as the ground shaking.

The same ground shaking is not equally harmful to all structures because they will have different natural frequencies and therefore, respond differently.

Response to Harmonic Ground Motions

- In **(a)**, the structure is in-phase with ground shaking, but have low amplitude.
- In **(b)**, the response of structure lags by a quarter-cycle to the ground shaking.
- In **(c)**, the mass of structure remains at the same place due to high inertial force, while the ground shakes. The structural response and ground shaking are completely out of phase.



Large dynamic response due to the effect of “Resonance”

Thank you