CE 809 - Structural Dynamics

Lecture 2: Free Vibration Response of SDF Systems

Semester – Fall 2020

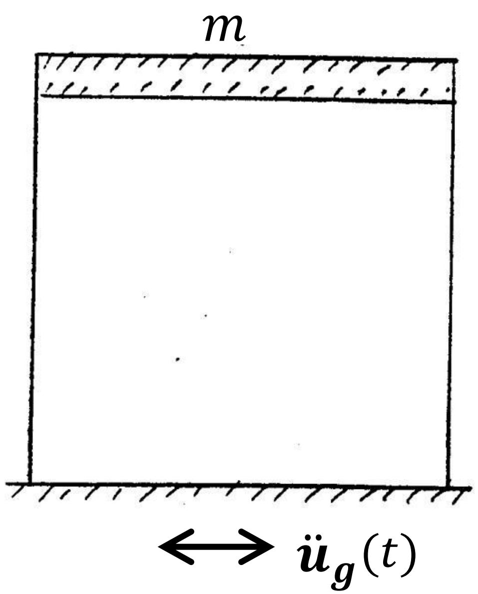
Dr. Fawad A. Najam
Department of Structural Engineering
NUST Institute of Civil Engineering (NICE)
National University of Sciences and Technology (NUST)
H-12 Islamabad, Pakistan
Cell: 92-334-5192533, Email: fawad@nice.nust.edu.pk

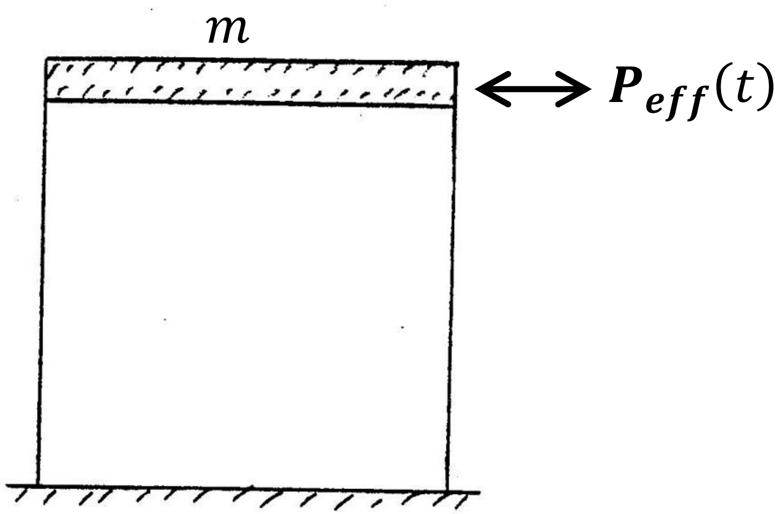
Prof. Dr. Pennung Warnitchai
Head, Department of Civil and Infrastructure Engineering
School of Engineering and Technology (SET)
Asian Institute of Technology (AIT)
Bangkok, Thailand

Equation of Motion of One-story Building

$$m\,\frac{d^2\boldsymbol{u}(t)}{dt^2}+c\,\frac{d\boldsymbol{u}(t)}{dt}+k\,\boldsymbol{u}(t)=-m\,\frac{d^2\boldsymbol{u_g}(t)}{dt^2}$$

$$m\,\frac{d^2\boldsymbol{u}(t)}{dt^2}+c\,\frac{d\boldsymbol{u}(t)}{dt}+k\,\boldsymbol{u}(t)\qquad\boldsymbol{P_{eff}}(t)$$





Fixed Base

The deformation $\boldsymbol{u}(t)$ of the structure due to ground acceleration $\ddot{\boldsymbol{u}}_{\boldsymbol{g}}(t)$ is identical to the deformation $\boldsymbol{u}(t)$ of the structure if its base were stationary and if it were subjected to an external force $\boldsymbol{P_{eff}}(t) = -m\ddot{\boldsymbol{u}}_{\boldsymbol{g}}(t)$.

Free Vibration Response of SDF Systems

Free vibration response: ***the motion of an SDF system with the applied force set equal to zero.***

Free vibration response in mathematical terms is ***the mathematical solution of the following homogeneous differential equation:***

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = 0 \qquad \textbf{\textit{Equation (1)}}$$

A Quick Review of Basic Mathematical Concepts

Solution form:

Consider a first-order differential equation

$$\frac{du(t)}{dt} + k\,u(t) = 0$$

$$\frac{du(t)}{dt} = -k\,u(t)$$

By separation of variables,

$$\frac{du(t)}{u(t)} = -k\,dt$$

Integrate both sides,

$$ln(u(t)) = -k\,t + c$$

Where c is an arbitrary constant.

By applying exponential operation,

$$e^{ln(u(t))} = u(t) \qquad e^{(-k\,t+c)} = e^{-k\,t}\,e^{c} = c_0\,e^{-k\,t}$$

The solution:

$$u(t) = c_0\,e^{-k\,t}$$

where c_0 is an arbitrary constant.

It can be shown that the solutions of higher order differential equations are also in this exponential form.

A Quick Review of Basic Mathematical Concepts

Superposition:

If a solution of a homogeneous linear differential equation is the multiplied by a constant, the resulting function is also a solution.

The sum of two solutions is also a solution.

Proof:

Let $\phi_1(t)$ and $\phi_2(t)$ be independent solutions of governing differential equation of an SDF system, such that

$$m\,\ddot{\phi}_1(t)\ + c\,\dot{\phi}_1(t)\ + k\,\phi_1(t)\ = 0$$

$$m\,\ddot{\phi}_2(t) \quad c\,\dot{\phi}_2(t)\ + k\,\phi_2(t)\ = 0$$

Substituting $c_1\phi_1(t)$ info the left-hand side of equation of motion (Eq (1)), we get

$$m\,(c_1\,\ddot{\phi}_1(t)) + c\,(c_1\,\dot{\phi}_1(t)\,) + k\,(c_1\,\phi_1(t)\,) =$$

$$c_1\,[m\,\ddot{\phi}_1(t) + c\,\dot{\phi}_1(t)\ + k\,\phi_1(t)\,] = c_1.0\ =\ 0$$

Hence $c_1\,\phi_1\,(t)$ is also a solution of the equation of motion (Eq (1)).

In similar manner, by a direct substitution of $c_1\,\phi_1(t)\ + c_2\,\phi_2(t)$ into the left-hand side of Eq (1), it can be shown that $c_1\,\phi_1(t)\ + c_2\,\phi_2(t)$ is also a solution of the equation of motion.

A Quick Review of Basic Mathematical Concepts

Initial Conditions

Consider $u(t) = c_1\,\phi_1(t) + c_2\,\phi_2(t)$ as a general solution of the governing equation of motion. Since the constants c_1 and c_2 can have any value, the general solution can represent ∞ different solutions.

Usually **initial conditions** are known and we are seeking for one specific solution that satisfies these initial conditions.

Example of initial conditions:

$u(0)$ and $\dot{u}(0)$ are the initial displacement and initial velocity of the SDF system.

Two conditions are needed because there are two unknown arbitrary constants to be specified.

$$\begin{aligned} u(0) &= c_1\,\phi_1(0) + c_2\,\phi_2(0) \\ \dot{u}(0) &= c_1\,\dot{\phi}_1(0) + c_2\,\dot{\phi}_2(0) \end{aligned}$$

$\phi_1(0)$, $\phi_2(0)$, $\dot{\phi}_1(0)$, $\dot{\phi}_2(0)$, $u(0)$ and $\dot{u}(0)$ all are known. Therefore c_1 and c_2 can be determined.

[For more details, see **Erwin Kreyszig's Advanced Engineering Mathematics**, John Wiley & Sons.]

Free Vibration Response of SDF Systems (continued)

Now consider the equation governing the free vibration of an SDF system:

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = 0 \quad \text{................ (1)}$$

Assuming that the solution of Eq (1) is in the exponential form:

$$u(t) = G\,e^{s\,t} \quad \text{................ (2)}$$

where G and s are constants.

Substituting this solution into the equation of motion (Eq (1)),

$$m\,(s^2\,G\,e^{s\,t}) + c\,(s\,G\,e^{s\,t}) + k\,(G\,e^{s\,t}) = 0$$

$$(m\,s^2 + c\,s + k)\,G\,e^{s\,t} = 0 \quad \text{................ (3)}$$

To have a non-zero solution of $u(t)$, the term $(m\,s^2 + c\,s + k)$ must be zero,

$$s^2 + \left(\frac{c}{m}\right)s + \left(\frac{k}{m}\right) = 0 \quad \text{................ (4)}$$

Case 1: Undamped Free Vibration Response

In this case, $c = 0$.

Introducing the notation

$$\omega = \sqrt{\frac{k}{m}}$$

The equation (4) becomes,

$$s^2 + \omega^2 = 0 \qquad \text{................ (5)}$$

Which has two solutions,

$$s = \pm\, i\, \omega \qquad \text{................ (6)}$$

Where $i = \sqrt{-1}$

Hence the general solution of $u(t)$ is

$$u(t) = G_1\, e^{i\,\omega\, t} + G_2\, e^{-i\,\omega\, t} \qquad \text{................ (7)}$$

Where G_1 and G_2 are arbitrary constants.

Case 1: Undamped Free Vibration Response (continued)

$$u(t) = G_1\, e^{i\,\omega\, t} + G_2\, e^{-\,i\,\omega\, t} \qquad \text{................ (7)}$$

Since there are two arbitrary constants, two initial conditions need to specified, i.e. $u(0)$ and $\dot{u}(0)$.

$$u(0) = G_1\, e^0 + G_2\, e^0 = G_1 + G_2$$

$$\dot{u}(0) = i\,\omega\, G_1\, e^0 - i\,\omega\, G_2\, e^0 = i\,\omega\, G_1 - i\,\omega\, G_2$$

Therefore,

$$\left.\begin{aligned} G_1 &= \frac{1}{2}\left(u(0) + \frac{\dot{u}(0)}{i\,\omega}\right) \\ G_2 &= \frac{1}{2}\left(u(0) - \frac{\dot{u}(0)}{i\,\omega}\right) \end{aligned}\right\} \qquad \text{................ (8)}$$

A Quick Review of Basic Mathematical Concepts

Taylor Series of e^x (expand around x = 0):

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for } -\infty < x < \infty$$

$$e^{i\,\omega\,t} = 1 + i\,\omega\,t + \frac{(i\,\omega\,t)^2}{2!} + \frac{(i\,\omega\,t)^3}{3!} + \cdots$$

$$e^{i\,\omega\,t} = 1 + i\,\omega\,t + (-1)\frac{(\omega\,t)^2}{2!} + (-1)\frac{i\,(\omega\,t)^3}{3!} + \cdots$$

$$e^{i\,\omega\,t} = \left\{1 - \frac{(\omega\,t)^2}{2!} + \cdots\right\} + i\left\{\omega\,t - \frac{(\omega\,t)^3}{3!} + \cdots\right\}$$

Taylor series of $\cos(\omega t)$ is

$$1 - \frac{(\omega\,t)^2}{2!} + \cdots$$

Similarly, the Taylor series of $\sin(\omega t)$ is

$$\omega\,t - \frac{(\omega\,t)^3}{3!} + \cdots$$

Therefore,

$$e^{\,i\,\omega\,t} = \cos(\omega t) + i\,\sin(\omega t)$$

This is called Euler's equation.

Case 1: Undamped Free Vibration Response (continued)

Introducing the Euler's equations:

$$e^{\pm i \omega t} = \cos(\omega t) \pm i \sin(\omega t) \qquad \text{................ (9)}$$

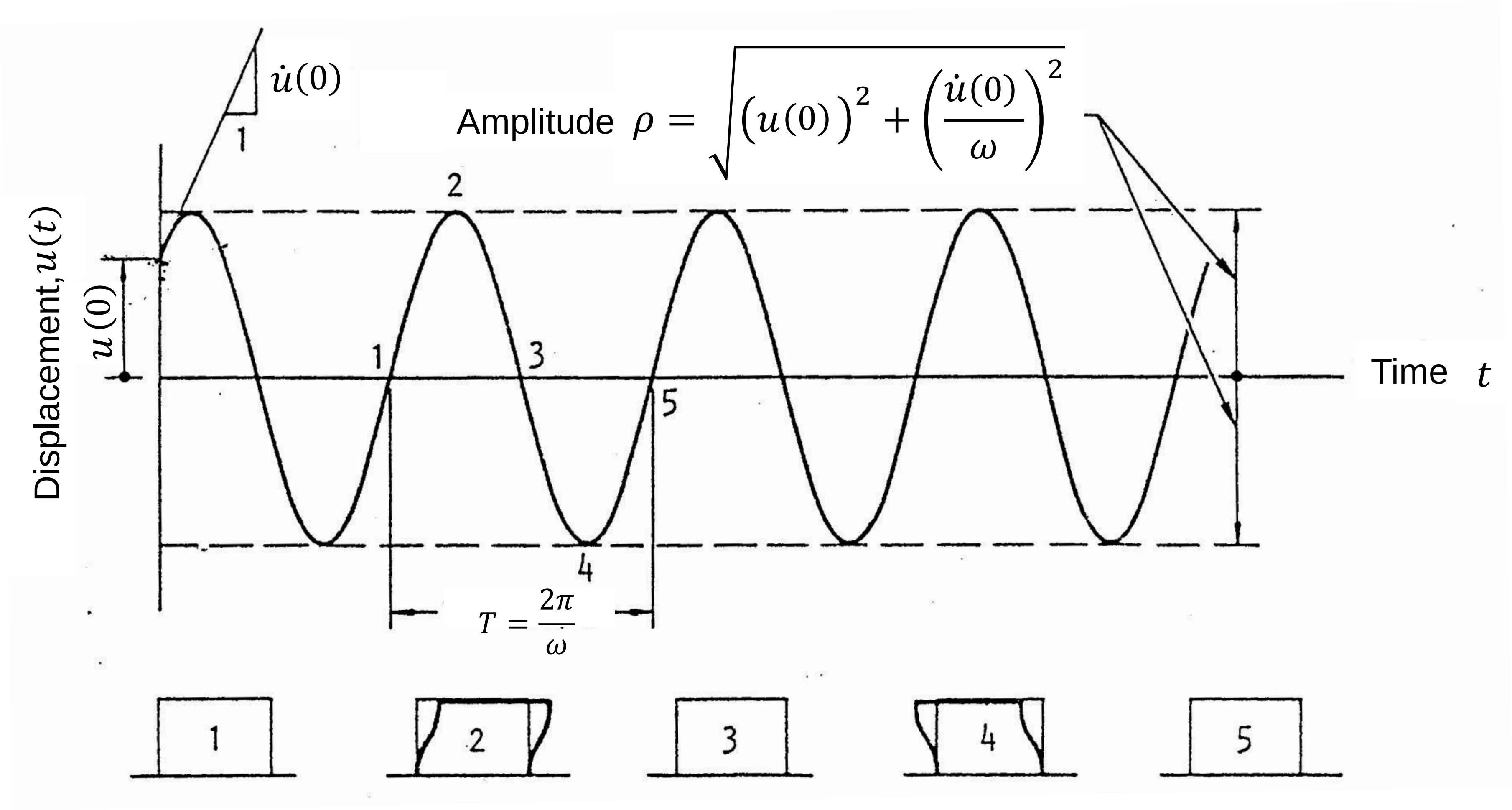
And the expressions for G_1 and G_2 (Eq (8)) into the solution (Eq (7)) , we obtain

$$u(t) = u(0) \cos(\omega t) + \frac{\dot{u}(0)}{\omega} \sin(\omega t) \qquad \text{................ (10)}$$

It is easy to verify that this equation is the solution of governing equation of motion by direct substitution.

Case 1: Undamped Free Vibration Response (continued)

$$u(t) = u(0)\cos(\omega t) + \frac{\dot{u}(0)}{\omega}\sin(\omega t)$$



Deformed position of structure corresponding to location 1, 2, 3, 4 and 5 on response-time plot

Case 1: Undamped Free Vibration Response (continued)

The structure vibrates in simple harmonic motion (or oscillation).

The amplitude of oscillation depends upon $u(0)$ and $\dot{u}(0)$. The above equation may be transformed into

$$u(t) = \rho \cos(\omega t - \theta) \qquad \ldots\ldots\ldots\ldots\ldots (11)$$

Where

$$\rho = \sqrt{\left(u(0)\right)^2 + \left(\frac{\dot{u}(0)}{\omega}\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{\dot{u}(0)}{\omega\, u(0)}\right) \qquad \ldots\ldots\ldots\ldots\ldots (12)$$

Case 1: Undamped Free Vibration Response (continued)

- The oscillation does not decay because the structure is undamped. The period of oscillation T is the time required for one cycle of free oscillation. For undamped structure,

$$T = \frac{2\pi}{\omega} = \frac{1}{f} \qquad \text{……………. (13)}$$

Where ω is the natural circular frequency,

f is the natural (cyclic) frequency (cycle/sec, Hz), and

T is the natural period (sec)

- This term "natural" is used to qualify each of the above quantities to emphasize the fact that these are “natural properties” of the structure.

- These properties are independent of the initial conditions.

Case 2: Damped Free Vibration Response

In this case $c \neq 0$; i.e. damping is present in the structure.

The solutions of $s^2 + \left(\frac{c}{m}\right)s + \left(\frac{k}{m}\right) = 0$ for this case are

$$s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \qquad \text{……………. (14)}$$

The characteristics of "s" depends upon the sign of the term $\left\{\left(\frac{c}{2m}\right)^2 - \omega^2\right\}$

Case 2 (a): The equation will have distinct real roots, if $\left(\frac{c}{2m}\right)^2 - \omega^2 > 0$

Case 2 (b): The equation will have complex conjugate root, if $\left(\frac{c}{2m}\right)^2 - \omega^2 < 0$

Case 2 (c): The equation will have real double roots, if $\left(\frac{c}{2m}\right)^2 - \omega^2 = 0$

Case 2 (b): Underdamped Systems ($c < 2\,m\,\omega$)

Let's define c_c: critical damping: $c_c \equiv 2\,m\,\omega$

Let's define ξ: critical damping ratio; $\xi \equiv \dfrac{c}{c_c} = \dfrac{c}{2\,m\,\omega}$ .... (15)

Hence, in underdamped systems, $0 < \xi < 1$

Rewriting the solution in terms of ξ, we get

$$s = -\xi\,\omega \pm \sqrt{(\xi\,\omega)^2 - \omega^2}$$

$$s = -\xi\,\omega \pm \sqrt{\omega^2(1-\xi^2)}\,\sqrt{-1}$$

$$s = -\xi\,\omega \pm i\,\omega_D \qquad \text{................ (16)}$$

Where $$\omega_D = \omega\,\sqrt{1-\xi^2} \qquad \text{................ (17)}$$

Case 2 (b): Underdamped Systems ($c < 2\,m\,\omega$) (continued)

Then the general solution of $u(t)$ is

$$u(t) = G_1\, e^{s_1\, t} + G_2\, e^{s_2\, t} = \left(G_1\; e^{(-\,\xi\,\omega\, t\,+\,i\,\omega_D\, t)} + G_2\; e^{(-\,\xi\,\omega\, t\,-\,i\,\omega_D\, t)}\right)$$

$$u(t)\; = e^{(-\,\xi\,\omega\, t)}\left(G_1\; e^{(-\,i\,\omega_D\, t)} + G_2\; e^{(-\,i\,\omega_D\, t)}\right) \qquad \text{…………….. (18)}$$

When the initial conditions of $u(0)$ and $\dot{u}(0)$ are introduced, the constants G_1 and G_2 can be evaluated, and after using Euler's equations we finally obtain,

$$u(t) = e^{(-\,\xi\,\omega\, t)}\left[\frac{\dot{u}(0)\; + u(0)\;\xi\,\omega}{\omega_D}\sin(\omega_D\; t) + u(0)\;\cos(\omega_D\; t)\right] \qquad \text{…………….. (19)}$$

Case 2 (b): Underdamped Systems ($c < 2\, m\, \omega$) (continued)

The response in above equation can also be presented as

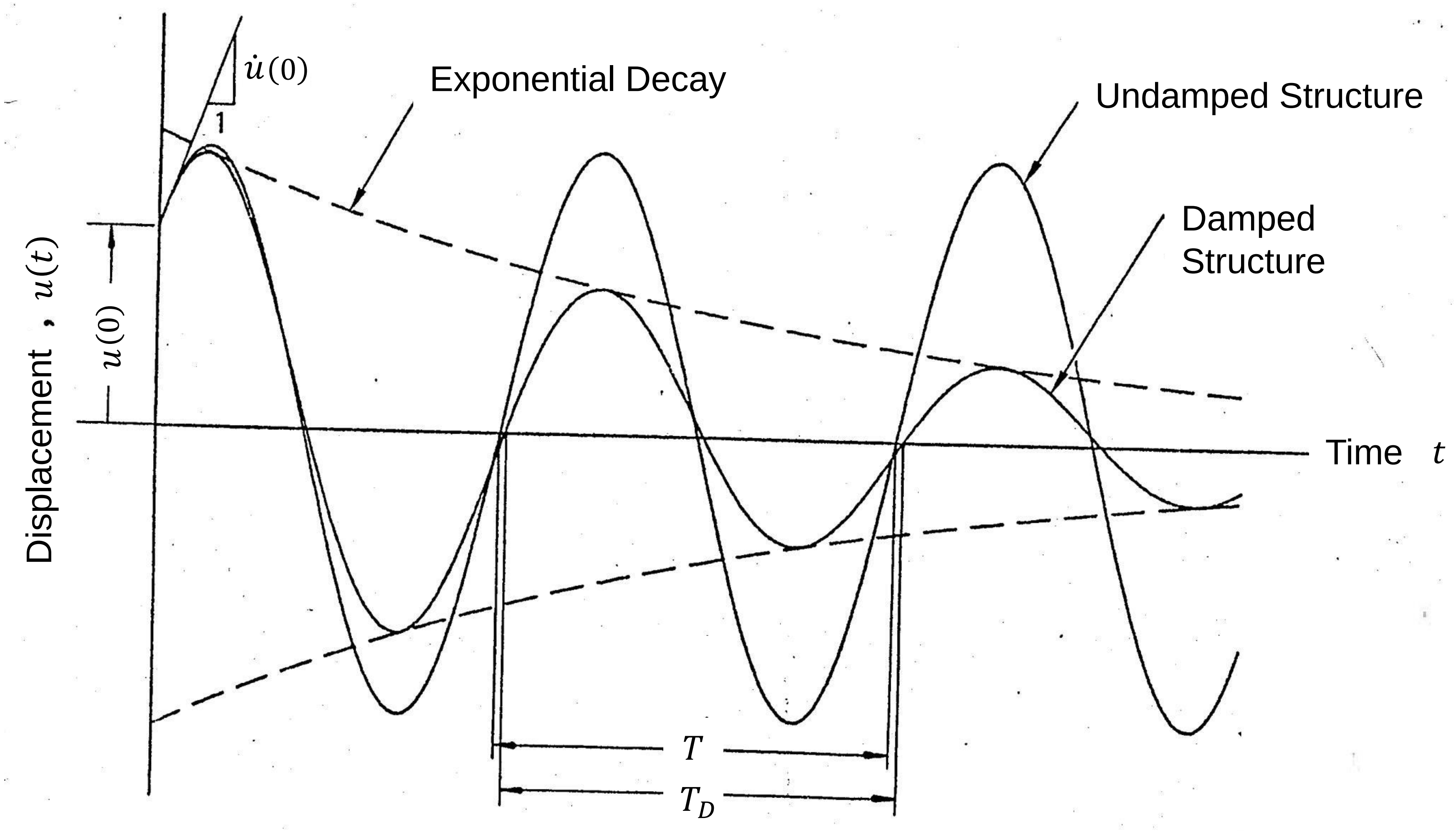
$$u(t) = e^{-\xi\, \omega\, t}\, \rho \cos(\omega_D\, t - \theta) \qquad \text{................ (20)}$$

Where

$$\left.\begin{aligned} \rho &= \sqrt{\left(\frac{\dot{u}(0) + u(0)\, \xi\, \omega}{\omega_D}\right)^2 + \left(u(0)\right)^2} \\ \theta &= \tan^{-1}\frac{\dot{u}(0) + u(0)\, \xi\, \omega}{\omega_D\, u(0)} \end{aligned}\right\} \qquad \text{................ (21 a, b)}$$

The equation (20) says that **the underdamped system in its free vibration stage will oscillate with circular frequency ω_D and with exponentially decreasing amplitude.**

Case 2 (b): Underdamped Systems ($c < 2\,m\,\omega$) (continued)

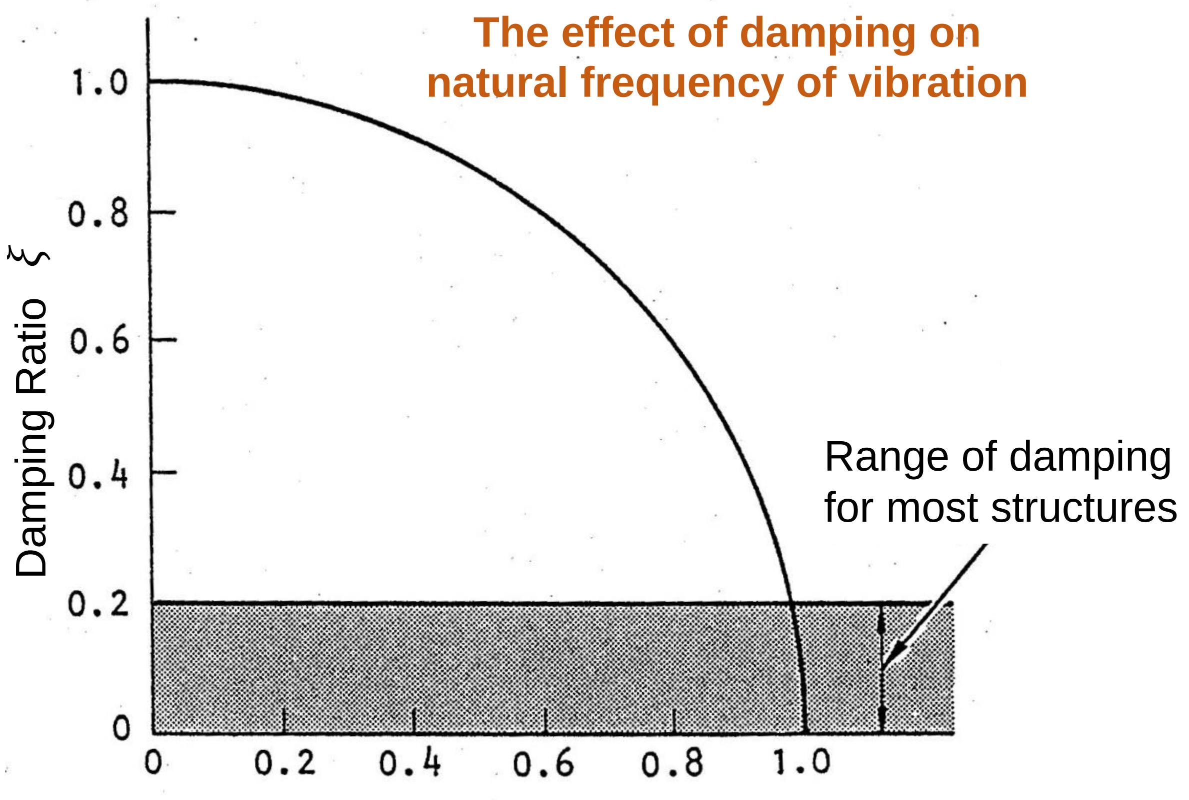


The effect of damping on free vibration

Effect of Damping on Free Vibration

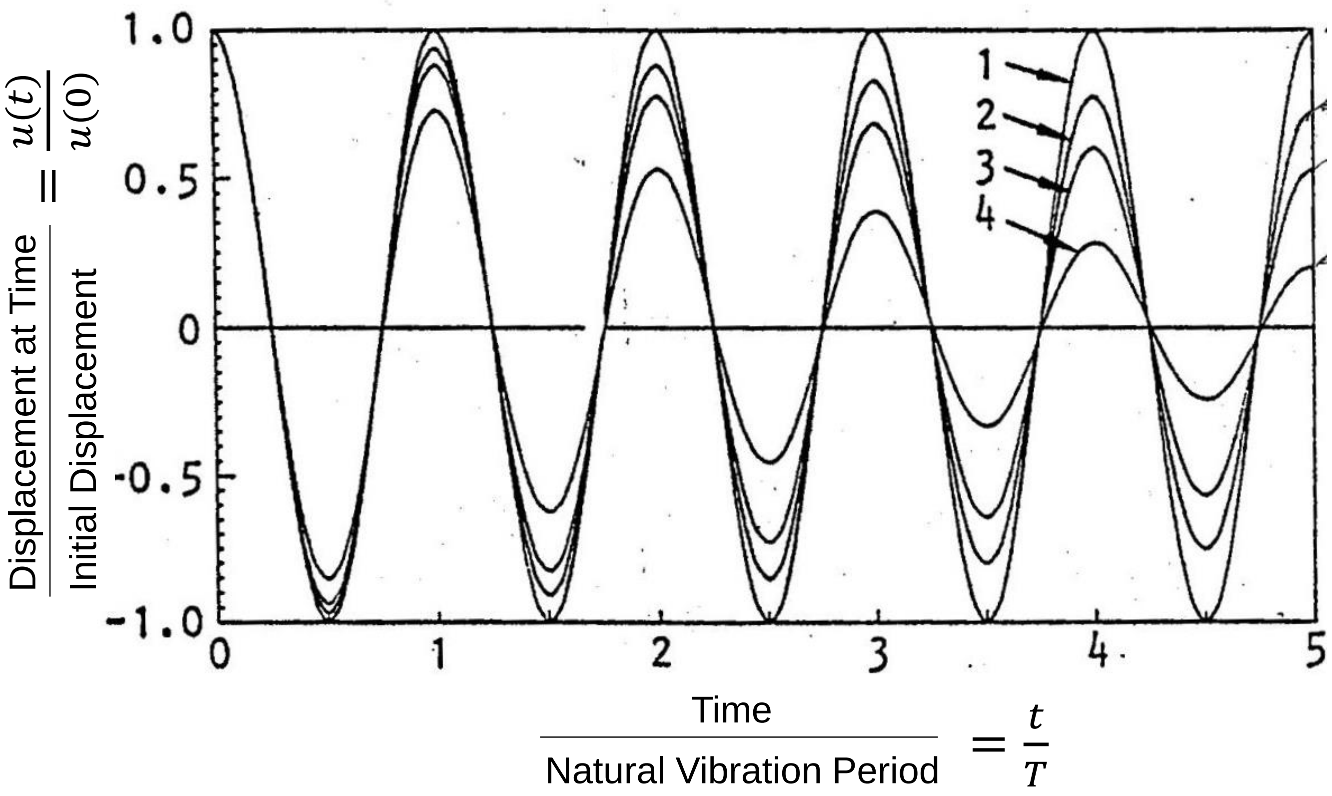
- In most structures, the critical damping ratio ξ is less than 0.2 and hence, $\omega_D = \omega$ and $T_D = T$.

- The rate of amplitude decay depends on ξ.



$$\frac{\text{Damped Natural Frequency}}{\text{Undamped Natural Frequency}} = \frac{\omega_D}{\omega}$$

Effect of Damping on Free Vibration



The effect of damping on free vibration. Curves 1, 2, 3 and 4 are for damping ratio 0, 1, 2 and 5 percent

Damping in Structures

- In seismic design of most structures, $\xi = 0.05$ is used.
- For tall buildings subjected to strong winds, we generally assume $\xi = 0.005 - 0.02$.
- For single cables, $\xi = 0.003 - 0.01$.

Type of Construction	**Typical Damping Ratios (ξ)**
Steel frame with welded connections and flexible walls	0.02
Steel frame with welded connections, normal floors and exterior cladding	0.05
Steel frame with bolted connections, normal floors and exterior cladding	0.1
Concrete frame with flexible internal walls	0.05
Concrete frame with flexible internal walls and exterior cladding	0.07
Concrete frame with concrete or masonry shear walls	0.1
Concrete or masonry shear wall	0.1
Wood frame and shear wall	0.15

Case 2 (c): Critical Damped Systems $(c = c_c = 2\,m\,\omega)$

In this case, $c = c_c = 2\,m\,\omega$ and $\xi = 1$. This will yield,

$$s = -\,\omega$$

The general solution of the governing equation of motion in this case will be of the form.

$$u(t) = G_1\ e^{s\,t} + t\ G_2\ e^{s\,t} = (G_1 + t\ G_2)\ e^{-\,\omega\,t})$$

The second term must contain t since the two roots of quadratic equation in s are identical.

$$\dot{u}(t) = -\omega\ (G_1 + t\ G_2)\ e^{-\,\omega\,t}) + G_2\ e^{-\,\omega\,t}$$

Case 2 (c): Critical Damped Systems $(c = c_c = 2\, m\, \omega)$

Using initial conditions $u(0)$ and $\dot{u}(0)$, the constants G_1 and G_2 can be determined as follows.

$$G_1 = u(0)$$

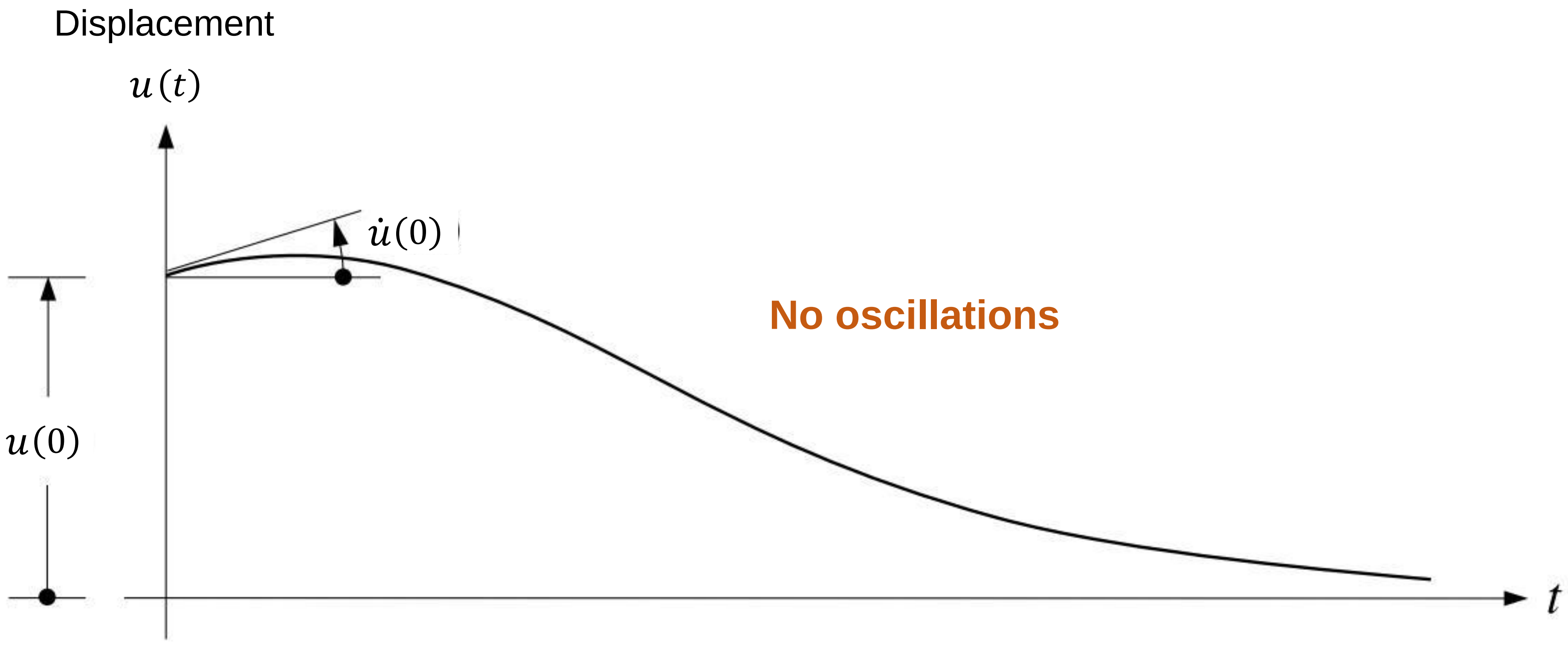
$$G_2 = \dot{u}(0) + \omega\, u(0)$$

The general solution will be,

$$u(t) = [u(0)\ (1 + \omega\, t) + \dot{u}(0)\ t]\, e^{-\omega\, t}$$

No oscillations. Critical damping just eliminated them.

Case 2 (c): Critical Damped Systems ($c = c_c = 2\,m\,\omega$)



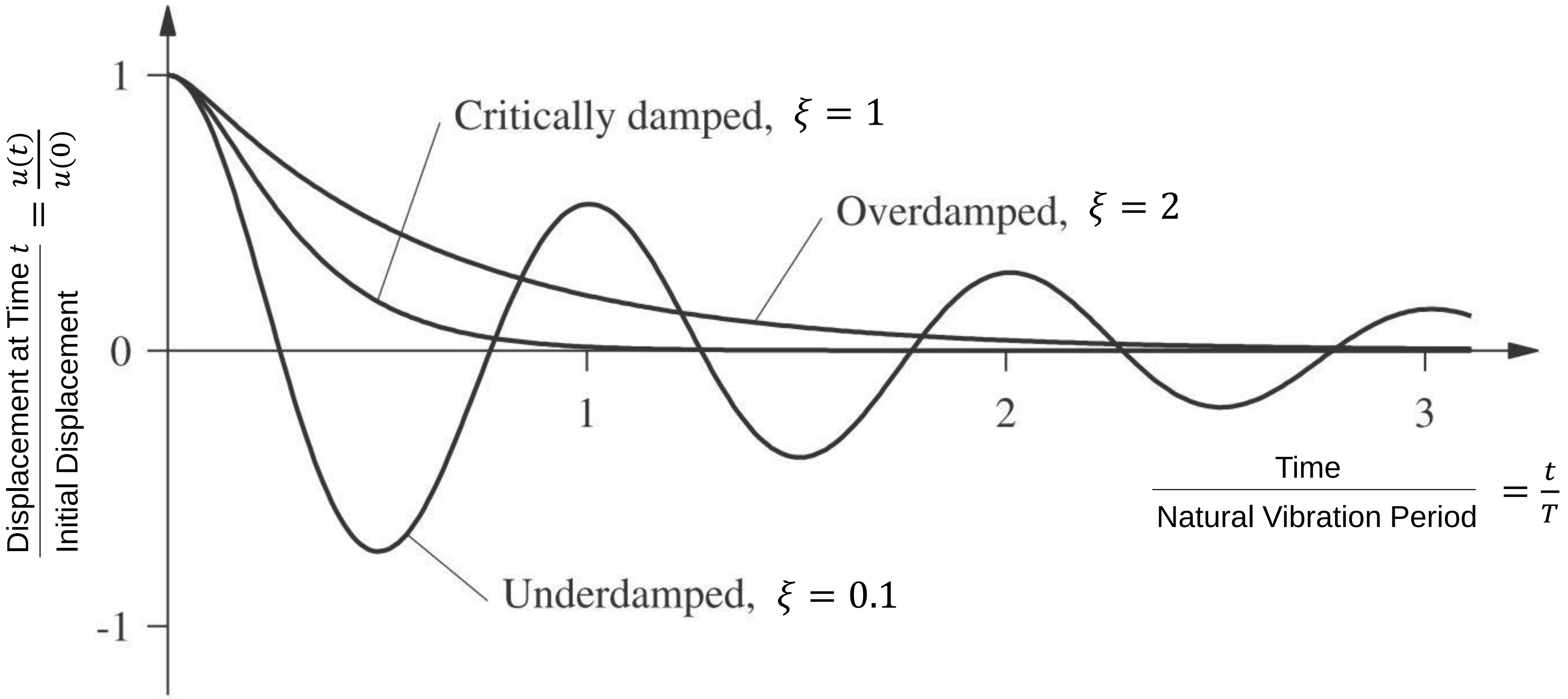
Free-vibration response with critical damping

(Clough and Penzien (2003) Dynamics of Structures, 3rd Edition).

Case 2 (a): Overdamped Systems ($c > c_c$)

- The response of an over-critically-damped system is similar to the motion of a critically-damped system.
- **Not encountered in practice**
- **No oscillations**

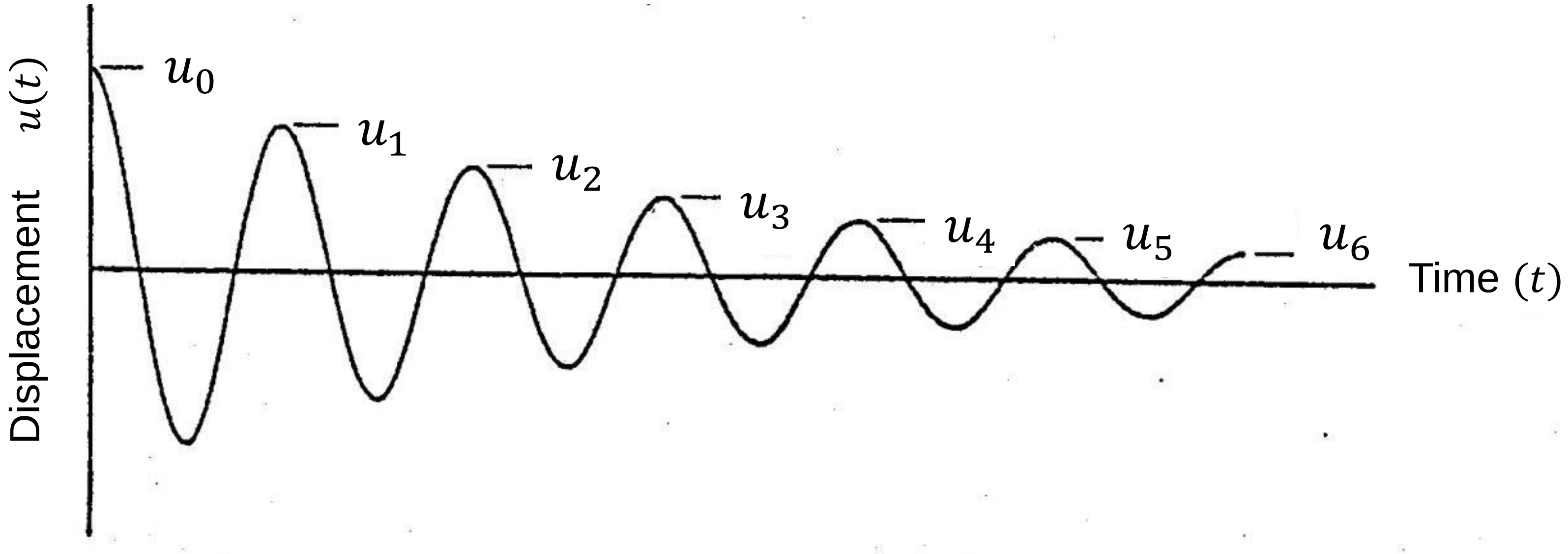
Summary



Free vibration of under-damped, critically damped, and over-damped systems

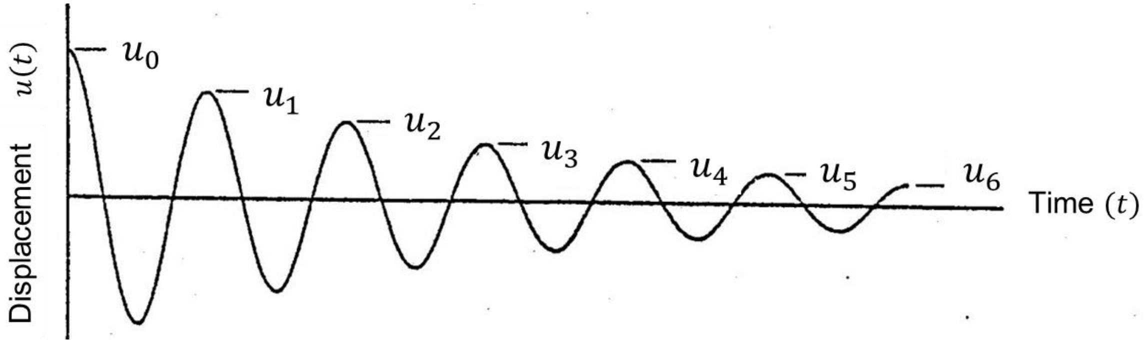
(Chopra (2012) Dynamics of Structures, 4th Edition)

Decay of Free Vibration Response



Measured displacement response from a free-vibration test

Free-vibration Tests



It can be shown that the ratio of any two successive peaks is

$$\frac{u_i}{u_{i+1}} = e^{\left(-2\,\pi\,\xi\,\frac{\omega}{\omega_D}\right)}$$

Taking the natural logarithm on both sides gives the logarithmic decrement δ, as follows.

$$\delta \equiv ln\left(\frac{u_i}{u_{i+1}}\right) = 2\,\pi\,\xi\,\frac{\omega}{\omega_D}$$

Hence for structure with low ξ,

$$\delta \approx 2\,\pi\,\xi$$

The above equation is very useful and can be used for the identification of ξ in existing structures.

Free-vibration Tests

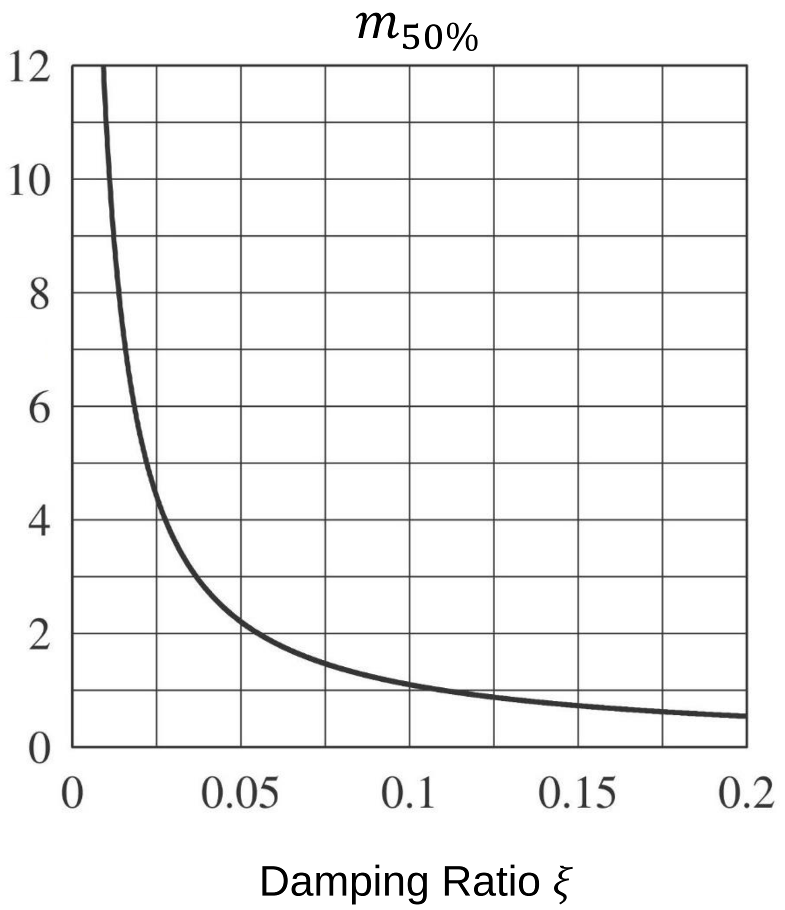
Sometimes it is more appropriate to consider the ratio $\frac{u_i}{u_{i+m}}$ where $m > 1$,

$$ln\left(\frac{u_i}{u_{i+m}}\right) = 2\,m\,\pi\,\xi\,\frac{\omega}{\omega_D}$$

$$\xi \approx \frac{1}{2\,m\,\pi}\,ln\left(\frac{u_i}{u_{i+1}}\right)$$

To determine the number of cycles elapsed for a 50% reduction in displacement amplitude ($m_{50\%}$), we obtain the following relation from the above equation.

$$m_{50\%} = \frac{0.11}{\xi}$$



The number of cycles required to reduce the free vibration amplitude by 50%

(Chopra (2012) Dynamics of Structures, 4th Edition)

Thank you