# CE 809 - Structural Dynamics

## Lecture 1: Formulation of a Mathematical Model of an SDF System

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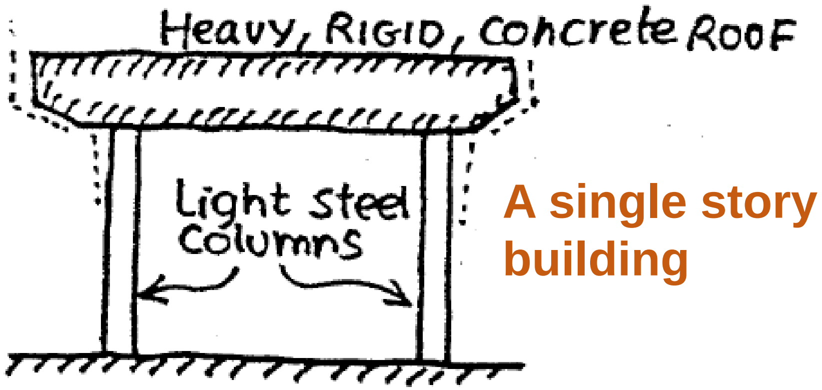
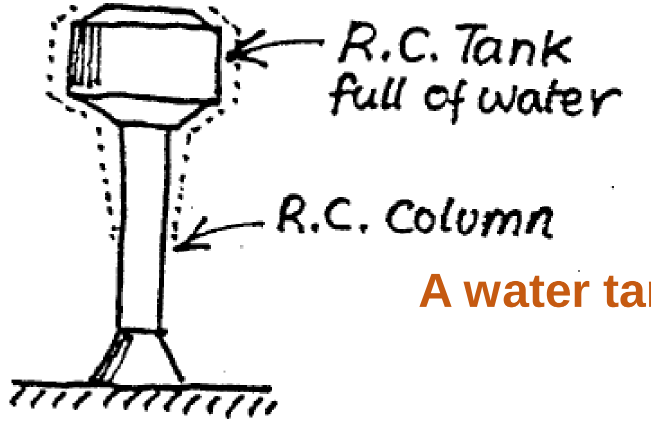
# Dynamics of Simple Structures

*Introduction to Basic Structural Dynamic Behaviors*

A simple structure = A structure that can be idealized as a concentrated mass "**m**" supported by a massless structure with stiffness "**k**" in the lateral direction.

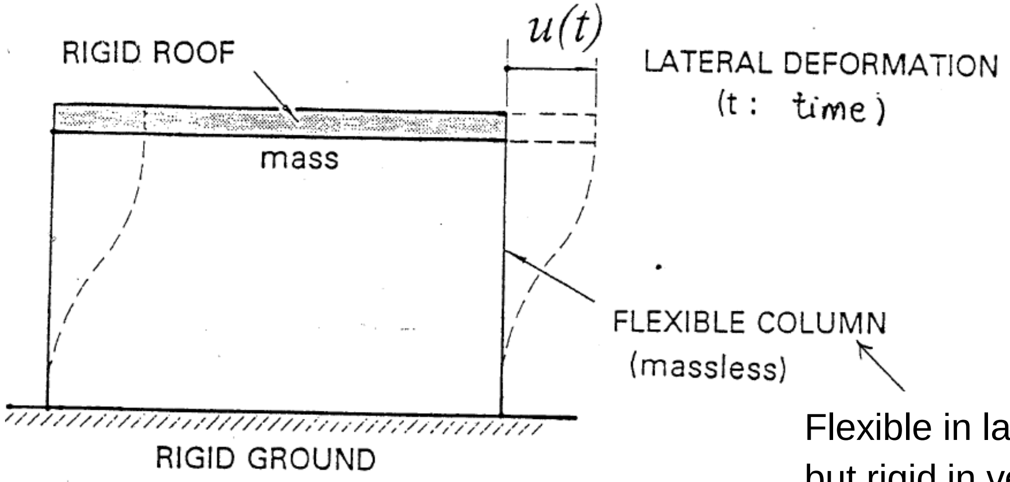



**A single story building**



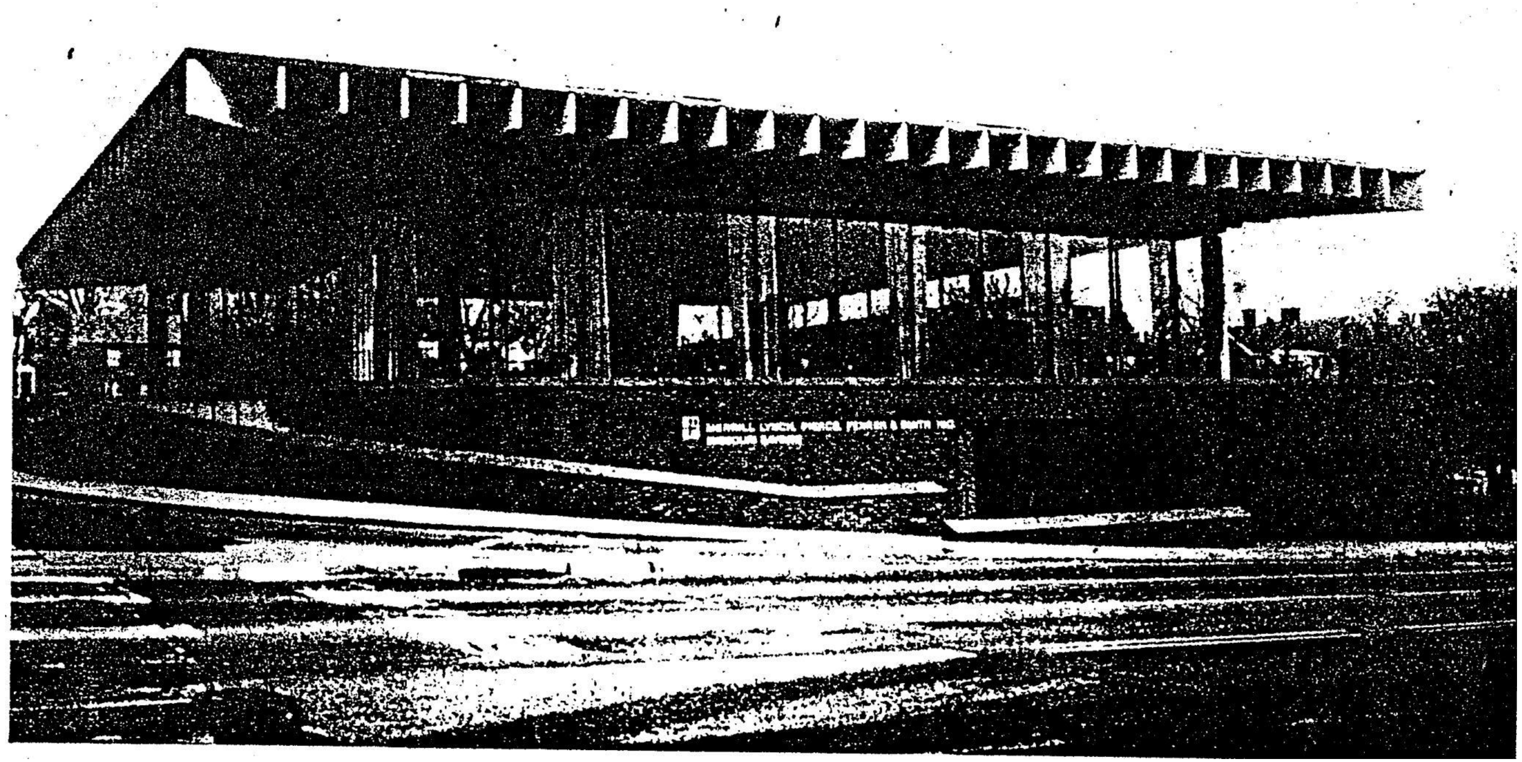

**A water tank**

## Idealization: A Single-Degree-of-Freedom System




Flexible in lateral direction but rigid in vertical direction

A simple structure which can be idealized as a single-degree-of-freedom (SDF) system



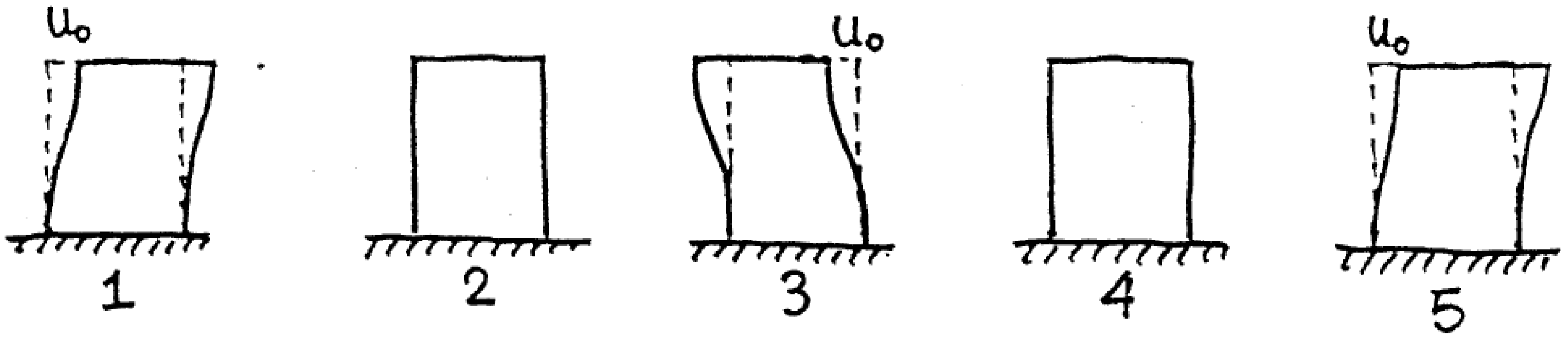
A one-story building. Most of the mass is concentrated at the roof level and the roof is essentially rigid compared to the lateral-force resisting system

Courtesy: G. W. Housner

Some examples of simple structures which can be idealized as single-degree-of-freedom (SDF) systems

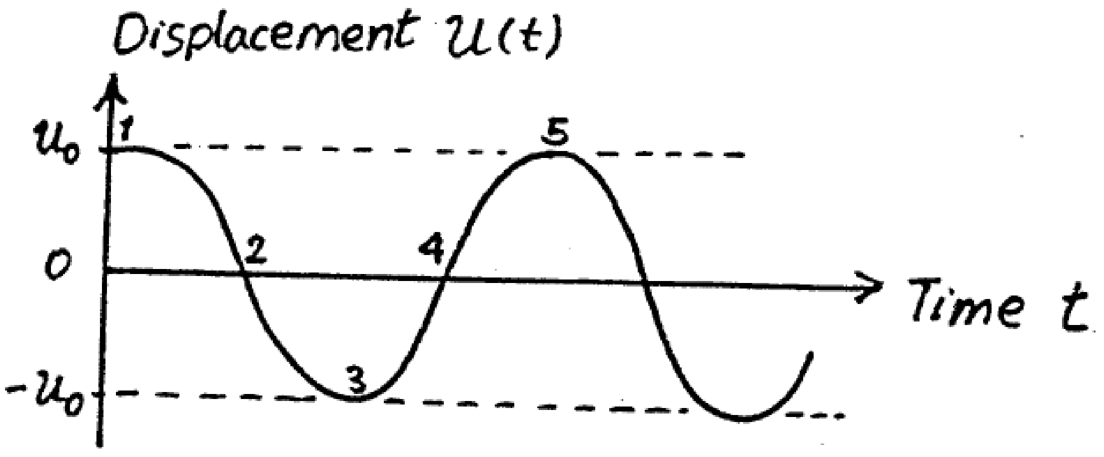
# Idealized Structural System

By this idealization, if the roof of a simple structure is displaced laterally by a distance $u_o$ and then released, the idealized structure will oscillate around its initial equilibrium configuration:




**The oscillation with amplitude $u_o$**

**The lateral displacement of roof as a function of time**



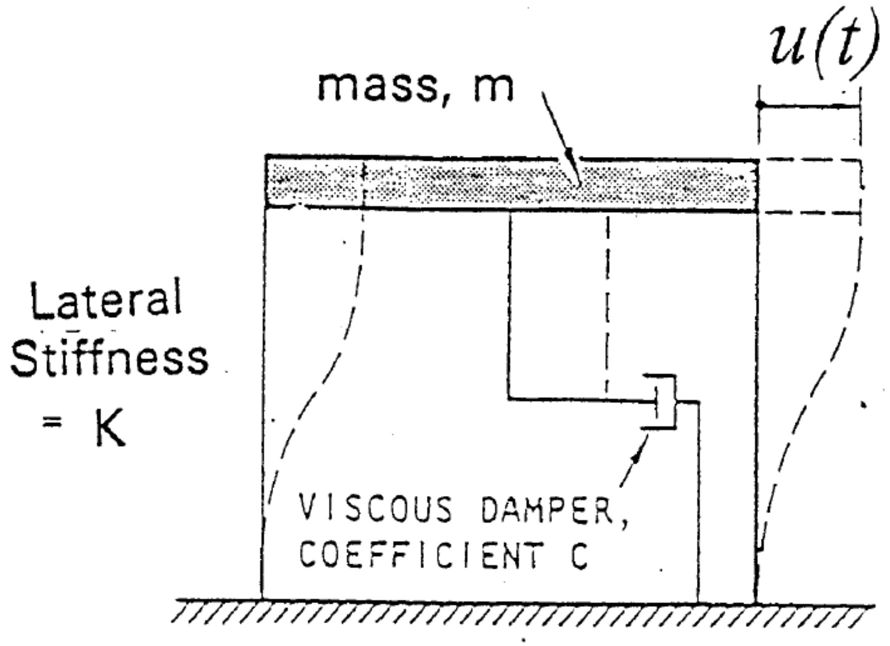

The oscillation will continue with the same amplitude $u_o$ and the idealized structure will never come to rest.

This is **an unrealistic response** because the actual structure will oscillate with decreasing amplitude and will eventually come to rest.

- To incorporate this feature into the idealized structure, an energy dissipating mechanism is required.

- Therefore, an energy absorbing element is introduced in the idealized structure: **the viscous damping element** (denoted by a dashpot).

- This simple structure is sometimes called **a Single-Degree-of-Freedom Structure.**




The functional elements of a single degree of freedom system

**Many basic concepts in structural dynamics can be understood by studying this simple structure.**

# Equation of Motion

- The motion of the idealized one-story structure caused by dynamic excitation is governed by an ordinary differential equation, called the **“equation of motion”**.

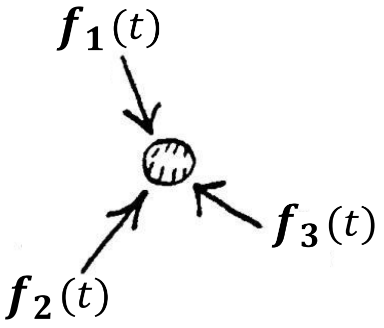
- Formulation of the equation is possibly the most important phase of the entire analysis procedure (and sometimes the most difficult phase).

- This equation can be determined using the following approaches:

    a) Direct Dynamic Equilibration Approach
    b) Principle of Virtual Work (Energy Approach)

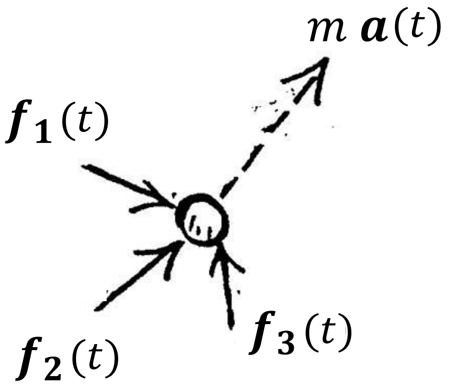
# Equation of Motion

- The Direct Equilibrium using **D'Alembert's Principle** will be employed in this lecture.



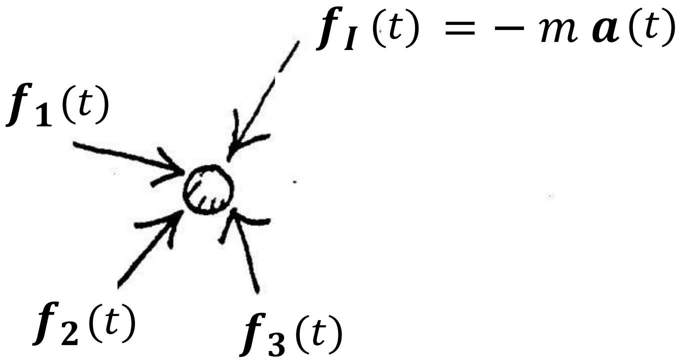

**Static Equilibrium**

$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t)=\boldsymbol{0}$$




**Newton's 2nd Law**

$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t)=m\,\boldsymbol{a}(t)$$




**D'Alembert's Principle**

$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t)+\boldsymbol{f}_I(t)=\boldsymbol{0}$$

A particle mass $m$ is subjected to a system of dynamic force vectors $\boldsymbol{f}_1(t),\boldsymbol{f}_2(t),\boldsymbol{f}_3(t)$

$\boldsymbol{a}(t)$ is the acceleration of the particle mass $m$

# Equation of Motion

Newton's 2nd law states that, "**The rate of change of momentum of any mass m is equal to the force acting on it**".

$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t) = \frac{d}{dt}\left(m\,\frac{d\boldsymbol{r}(t)}{dt}\right) = m\frac{d^2\boldsymbol{r}(t)}{dt^2} = m\,\boldsymbol{a}(t)$$

D'Alembert's concept states that "**A mass develops an inertia force in proportion to its acceleration and opposing it**".

$$\boldsymbol{f}_I(t) = -\,m\,\boldsymbol{a}(t)$$

**Newton's 2nd law:**

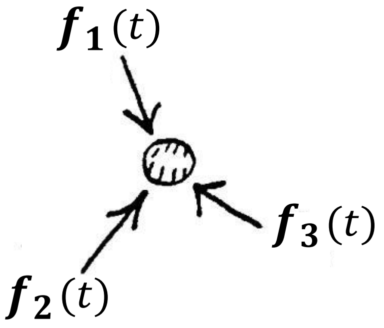
$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t)+\boldsymbol{f}_I(t) = \boldsymbol{0}$$

*All dynamic forces (including the inertia force) are in equilibrium:*
**Dynamic Equilibrium**

This is a very convenient concept in structure dynamics because its permits equations of motion to be expressed as "**equations of dynamic equilibrium**".

# Equation of Motion

- The Direct Equilibrium using **D'Alembert's Principle** will be employed in this lecture.




**Static Equilibrium**

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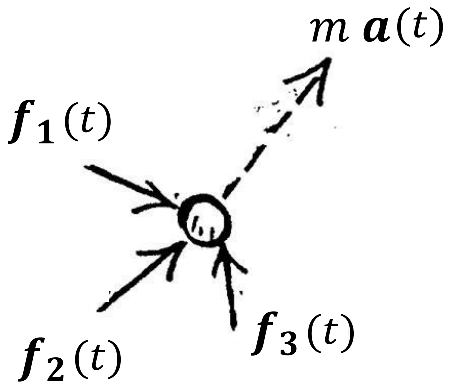
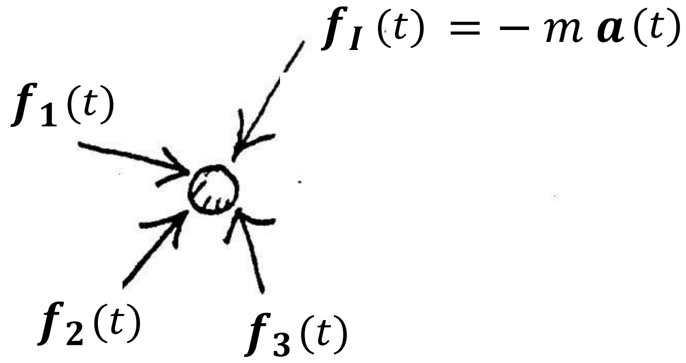



**Newton's 2nd Law**

$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t)=m\,\boldsymbol{a}(t)$$

**Dynamic Equilibrium**



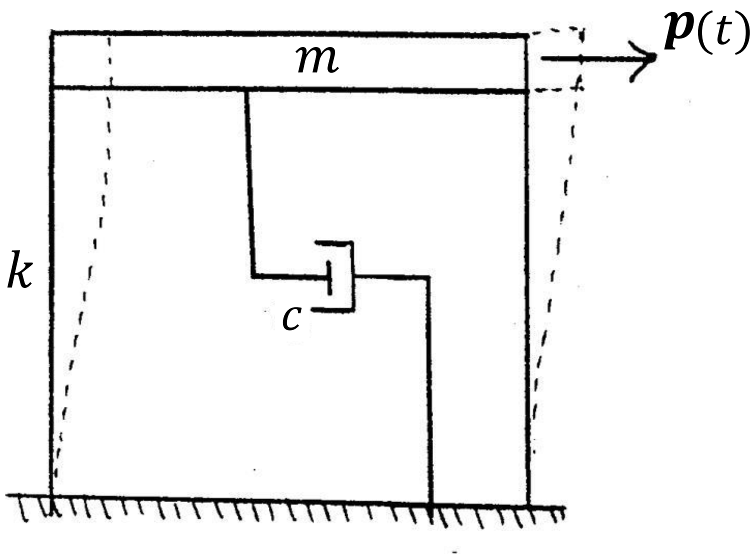

**D'Alembert's Principle**

$$\boldsymbol{f}_1(t)+\boldsymbol{f}_2(t)+\boldsymbol{f}_3(t)+\boldsymbol{f}_I(t)=\boldsymbol{0}$$

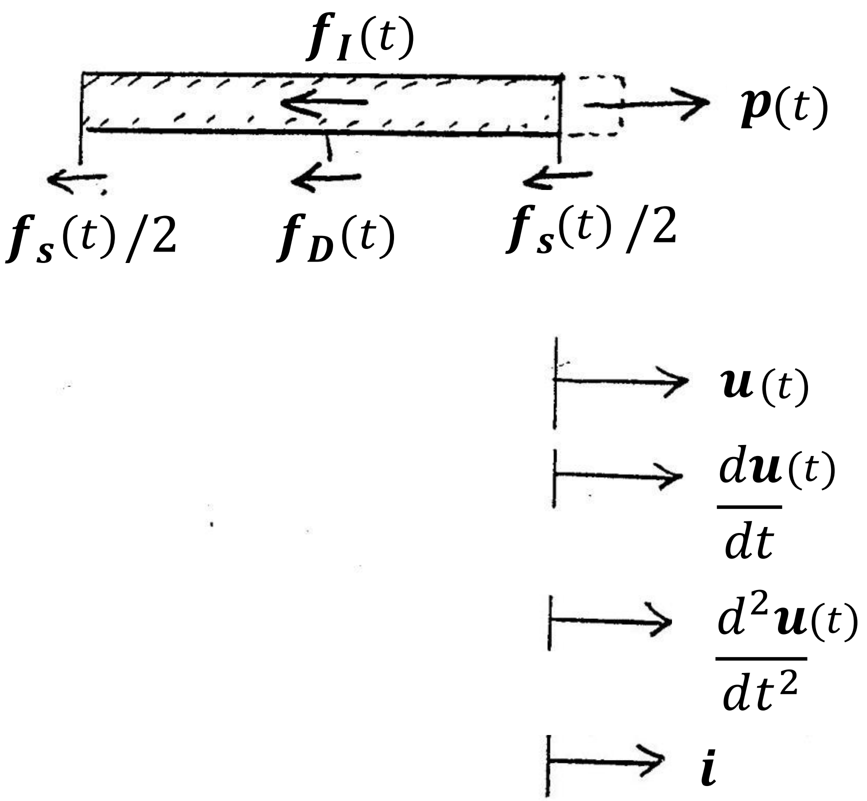
A particle mass $m$ is subjected to a system of dynamic force vectors $\boldsymbol{f}_1(t), \boldsymbol{f}_2(t), \boldsymbol{f}_3(t)$

$\boldsymbol{a}(t)$ is the acceleration of the particle mass $m$

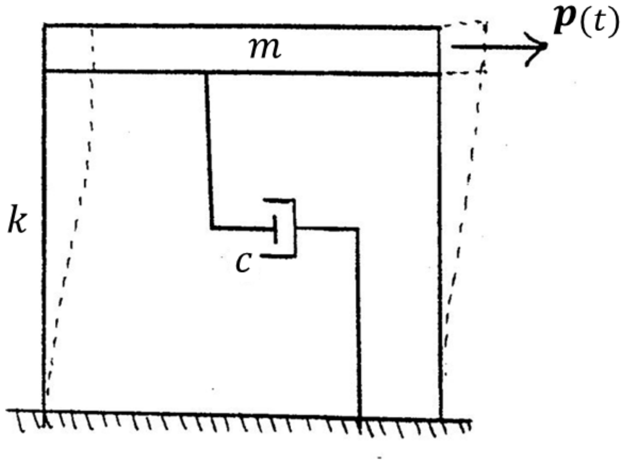
# Equation of Motion of One-story Building Subjected to Dynamic Force

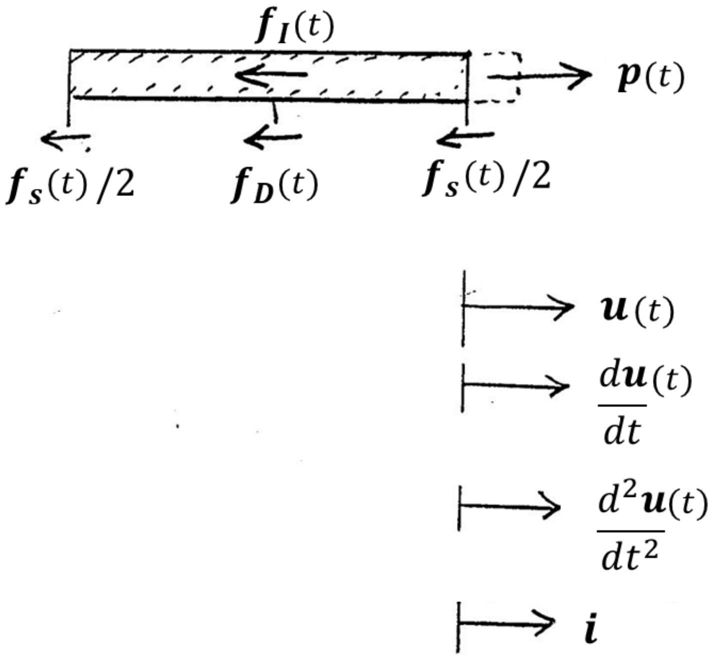



Free-Body Diagram




At any instantaneous time, the mass $m$ is under the action of **four types of dynamic forces**.



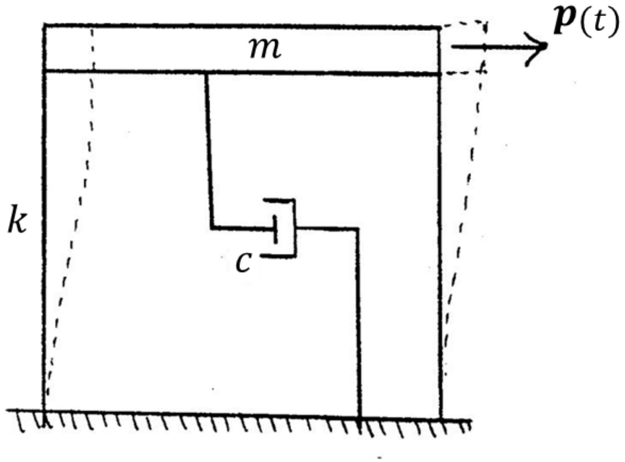

1. External dynamic force: $\boldsymbol{p}(t)$
2. Inertia force:

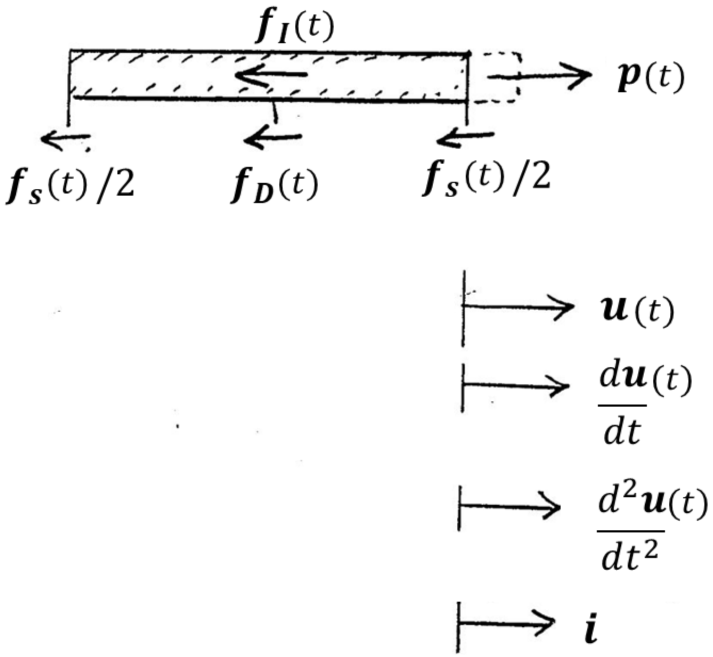
$$\boldsymbol{f_I}(t) = -m\frac{d^2\boldsymbol{u}(t)}{dt^2}$$

3. Elastic force:

$$\boldsymbol{f_s}(t) = -k\,\boldsymbol{u}(t)$$

where $k$ is the lateral stiffness of the two columns combined. The negative sign means that the forces is always in the opposite direction to the structural deformation (this is to bring the structure back to its neutral position).




4. Damping force:

$$\boldsymbol{f_D}(t) = -c\,\frac{d\boldsymbol{u}(t)}{dt} = c\,\dot{\boldsymbol{u}}(t)$$

where $c$ is the damping coefficient of viscous damper. The units of $c$ are force×time/length. The negative sign means that the damping force is always in the opposite direction to velocity, hence it always dissipates energy.

By the application of D'Alembert's principle, the sum of all four forces must be zero.

$$\boldsymbol{f_1}(t)+\boldsymbol{f_2}(t)+\boldsymbol{f_3}(t)+\boldsymbol{p}(t) = \boldsymbol{0}$$

Or

$$m\,\frac{d^2\boldsymbol{u}(t)}{dt^2}+c\,\frac{d\boldsymbol{u}(t)}{dt}+k\,\boldsymbol{u}(t) \qquad \boldsymbol{p}(t)$$

The vector can be converted to scalar function by

$$\boldsymbol{u}(t) = u(t)\ \boldsymbol{i}$$

$$\frac{d\boldsymbol{u}(t)}{dt}=\frac{du(t)}{dt}.\boldsymbol{i}$$

$$\frac{d^2\boldsymbol{u}(t)}{dt^2}=\frac{d^2u(t)}{dt^2}.\boldsymbol{i}$$

$$\boldsymbol{p}(t) = p(t).\boldsymbol{i}$$

$p$ and $u$ are a function of time. $\boldsymbol{i}$ is a unit length base vector.

Hence, the equation of motion in scalar form is

$$m\,\frac{d^2u(t)}{dt^2}+c\,\frac{du(t)}{dt}+k\,u(t)=p(t)$$

**This is a second-order linear (ordinary) differential equation.**

# Problem Statement

**Given:**

a) The mass of the system ($m$),

b) Applied dynamic load $\boldsymbol{p}(t)$,

c) Lateral stiffness of the system ($k$), and

d) The damping coefficient of the system ($c$)

**Determine:**

The displacement of the system $\boldsymbol{u}(t)$

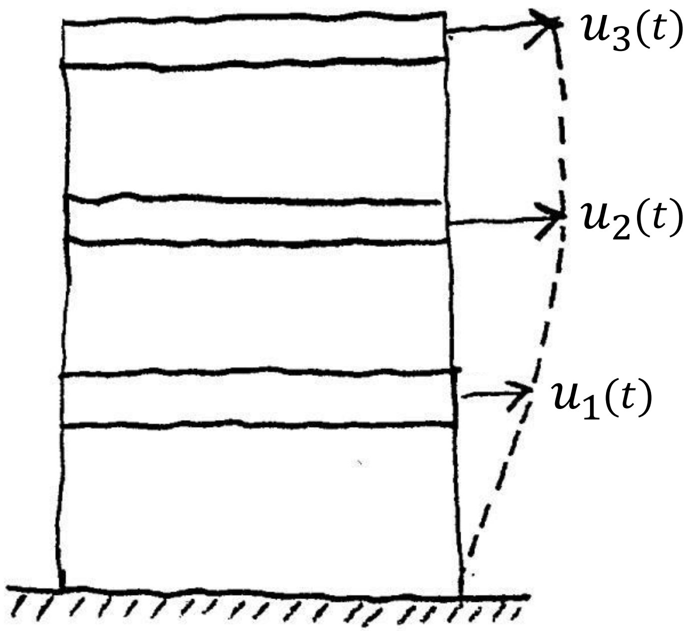
The other response quantities (e.g. the response velocity $\frac{d\boldsymbol{u}(t)}{dt}$, response acceleration $\frac{d^2\boldsymbol{u}(t)}{dt^2}$, base shear, overturning moment etc.) can be subsequently derived from $\boldsymbol{u}(t)$.

# Multi-Degree-of-Freedom Structures

- The example (idealized one-story) structure described earlier is a single-degree-of-freedom system because its motion can be completely describe by only one scalar function – $u(t)$.

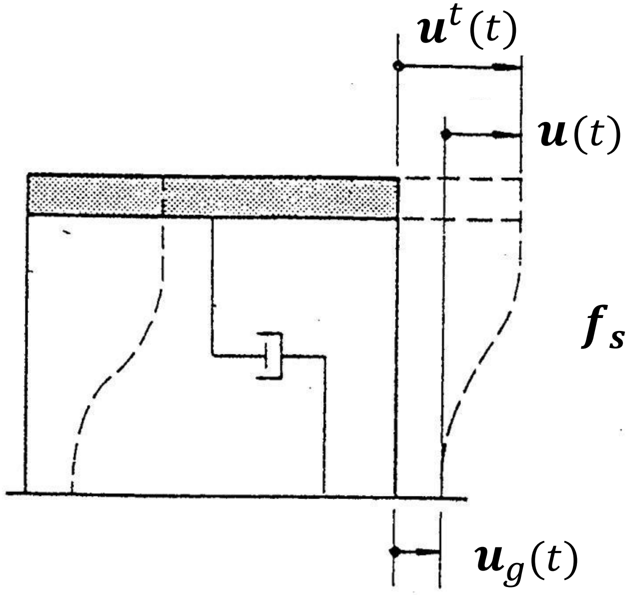
- A 3-story building is a three-degree-of-freedom system because at least 3 response functions ($u_1(t)$ $u_2(t)$ $u_3(t)$) are required to completely describe the overall motion of this structure.

- The dynamics of multi-degree-of-freedom systems will be covered in detail later.

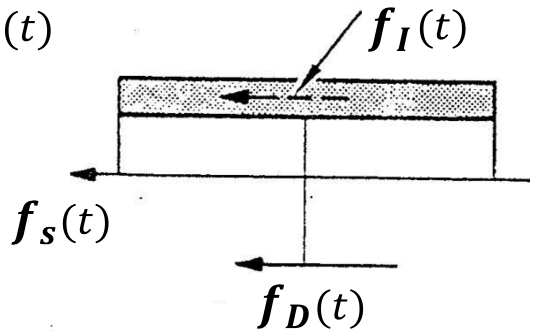



A three-degree-of-freedom system

# Equation of Motion of One-story Building subjected to Earthquake




Idealized structure




Free-body diagram

- Consider a case when an SDF system is subjected to a lateral ground displacement $u_g(t)$ .
- This represents a simplified earthquake excitation (i.e. the ground motion is assumed to be a one-dimensional lateral motion).
- There is no external force applied to this SDF system.

Let's denote the ground displacement, ground velocity and ground acceleration as

$$\boldsymbol{u}_g(t), \qquad \frac{d\boldsymbol{u}_{\boldsymbol{g}}(t)}{dt}, \qquad \frac{d^2\boldsymbol{u}_{\boldsymbol{g}}(t)}{dt^2}$$

The total displacement at the roof is defined by $\boldsymbol{u}^t(t)$, where

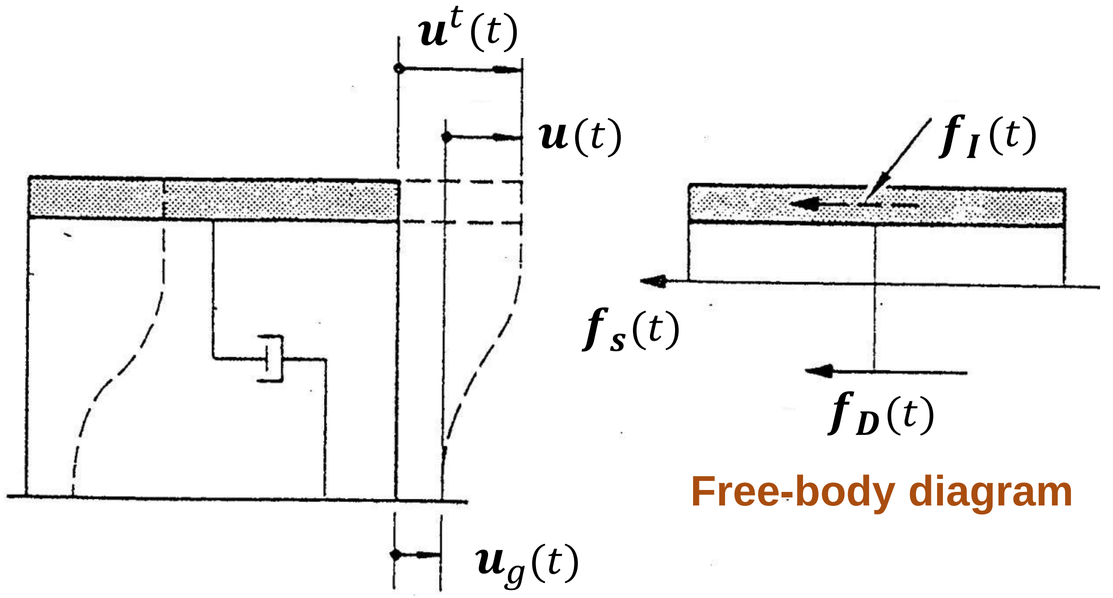
$$\boldsymbol{u}^t(t) = \boldsymbol{u}_g(t) + \boldsymbol{u}(t)$$

There are three dynamic forces acting on the roof mass:

1. Elastic force $\boldsymbol{f}_s(t) = -k\,\boldsymbol{u}(t)$

2. Damping force $\boldsymbol{f}_D(t) = -c\dfrac{d\boldsymbol{u}(t)}{dt}$

Each of these forces is a function of "relative" motion, not the absolute (or total) motion.

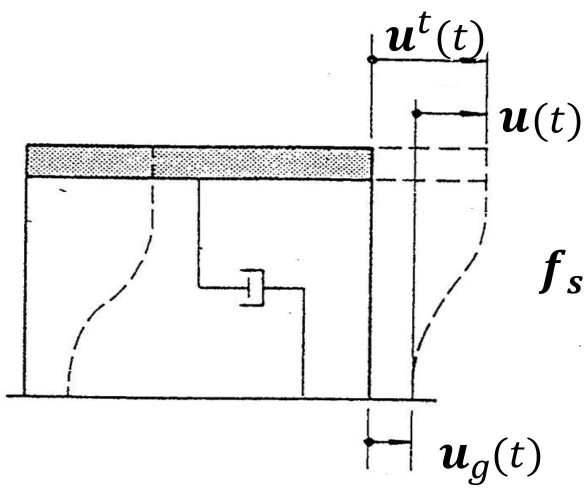



Idealized structure

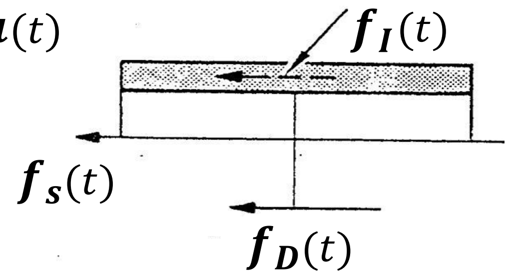
Each of these forces is a function of “relative” motion, and not the absolute (or total) motion. However the mass undergoes an acceleration of $\dfrac{d^2\boldsymbol{u}^t(t)}{dt^2}$

Therefore

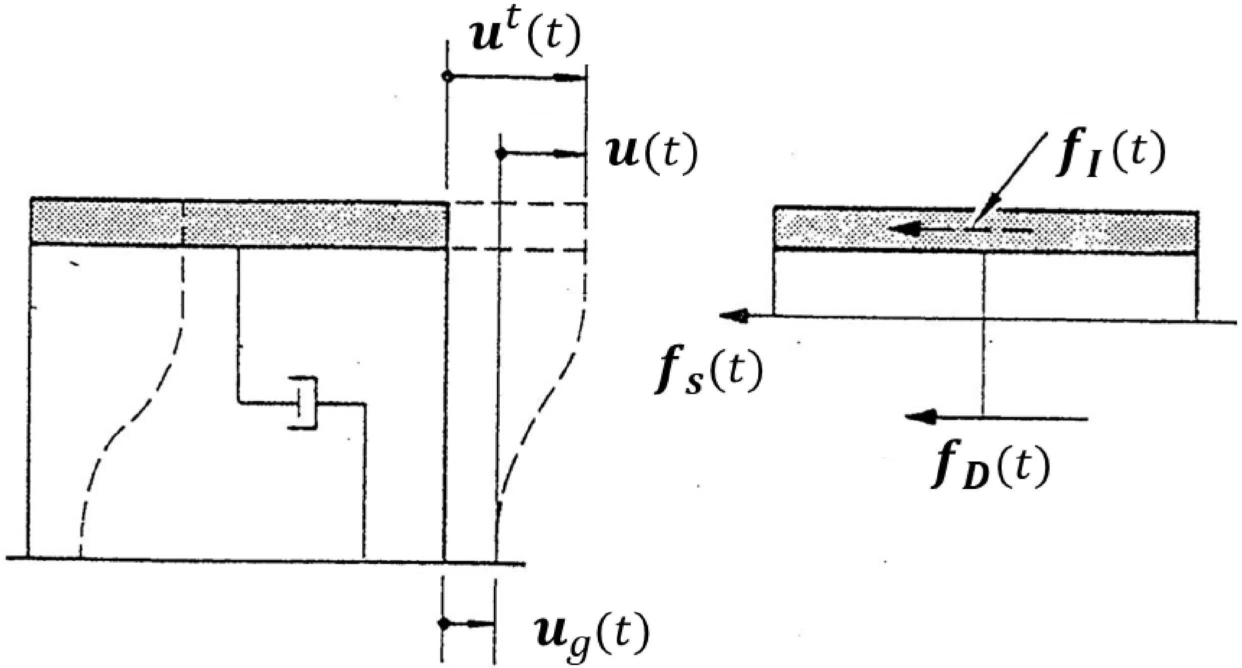
3. Inertia force $\boldsymbol{f_I}(t) = -m\dfrac{d^2\boldsymbol{u}^t(t)}{dt^2} = -m\dfrac{d^2\boldsymbol{u_g}(t)}{dt^2} - m\dfrac{d^2\boldsymbol{u}(t)}{dt^2}$



Idealized structure



Free-body diagram




Applying the D'Alembert's dynamic equilibrium to this case, we get,

$$m \frac{d^2\boldsymbol{u}(t)}{dt^2} + c \frac{d\boldsymbol{u}(t)}{dt} + k\,\boldsymbol{u}(t) = -m \frac{d^2\boldsymbol{u}_g(t)}{dt^2}$$

In scalar form,

$$m \frac{d^2u(t)}{dt^2} + c \frac{du(t)}{dt} + k\,u(t) = -m \frac{d^2u_g(t)}{dt^2}$$

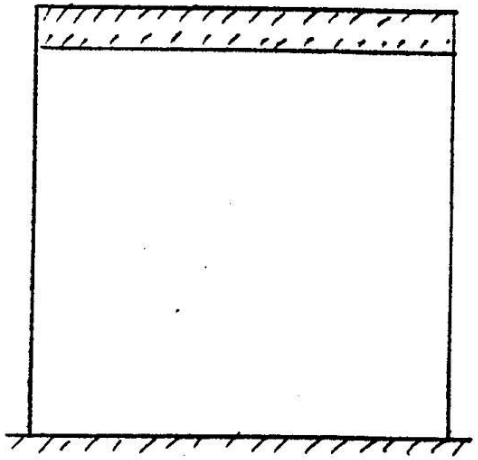
This equation of motion is the governing equation of structural deformation $\boldsymbol{u}(t)$, when the structure is subjected to ground acceleration $\frac{d^2\boldsymbol{u_g}(t)}{dt^2}$ .

# Equation of Motion of One-story Building subjected to Earthquake

$$m\,\frac{d^2\boldsymbol{u}(t)}{dt^2} + c\,\frac{d\boldsymbol{u}(t)}{dt} + k\,\boldsymbol{u}(t) = -m\,\frac{d^2\boldsymbol{u_g}(t)}{dt^2}$$

$$m\,\frac{d^2\boldsymbol{u}(t)}{dt^2} + c\,\frac{d\boldsymbol{u}(t)}{dt} + k\,\boldsymbol{u}(t) \qquad \boldsymbol{P_{eff}}(t)$$

$m$



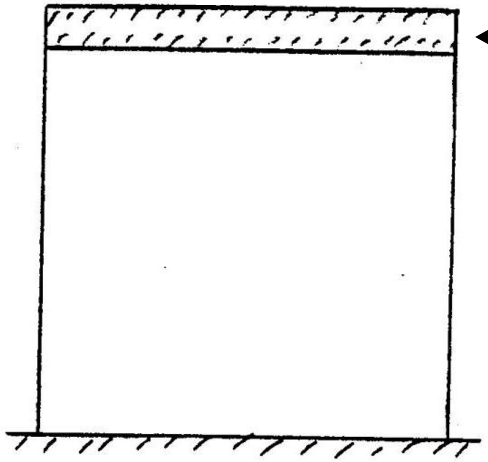
$\longleftrightarrow\ \boldsymbol{\ddot{u}_g}(t)$

Identical

=

response $\boldsymbol{u}(t)$

$m$



$\longleftrightarrow\ \boldsymbol{P_{eff}}(t)$

Fixed Base

Where

$$\boldsymbol{P_{eff}}(t) = -m\,\frac{d^2\boldsymbol{u_g}(t)}{dt^2}$$

$$= -m\,\boldsymbol{\ddot{u}_g}(t)$$

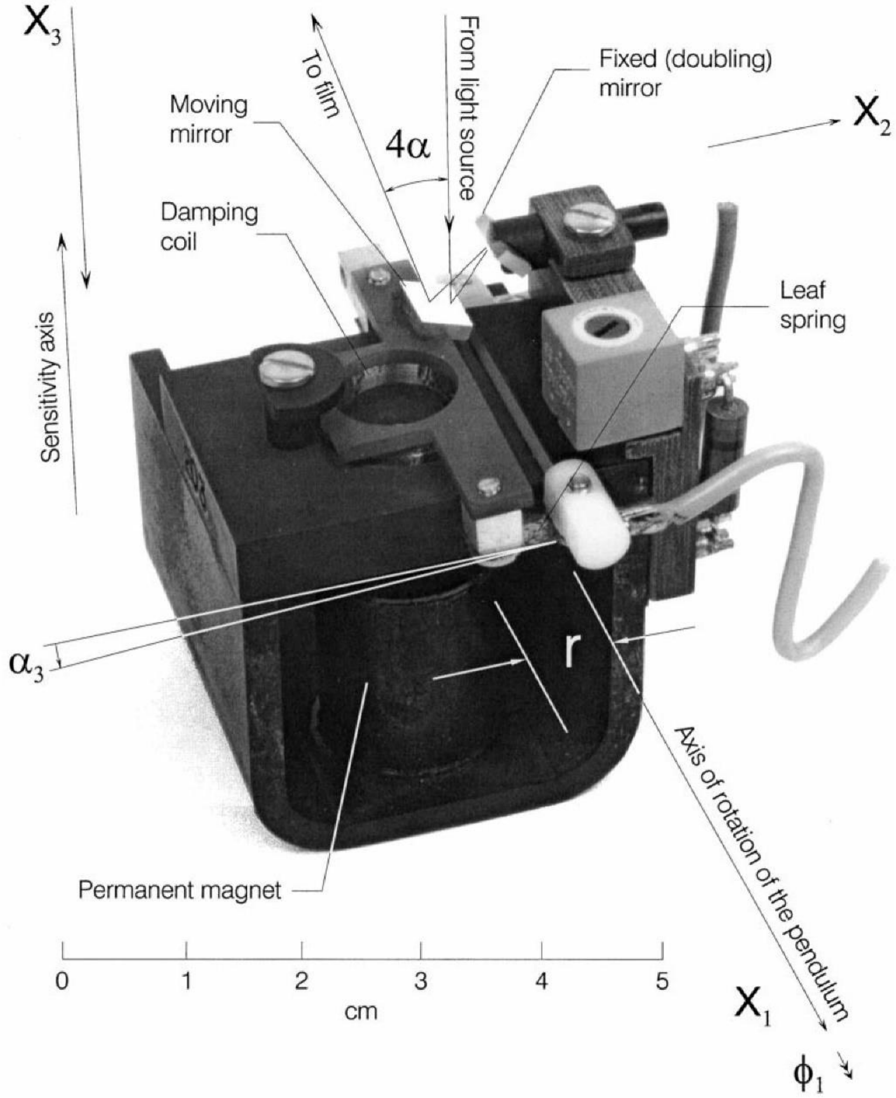
The deformation $\boldsymbol{u}(t)$ of the structure due to ground acceleration $\boldsymbol{\ddot{u}_g}(t)$ is identical to the deformation $\boldsymbol{u}(t)$ of the structure if its base were stationary and if it were subjected to an external force $\boldsymbol{P_{eff}}(t) = -m\boldsymbol{\ddot{u}_g}(t)$.

# Kinemetrics Altus K2 Strong Motion Accelerograph System

**Applications**

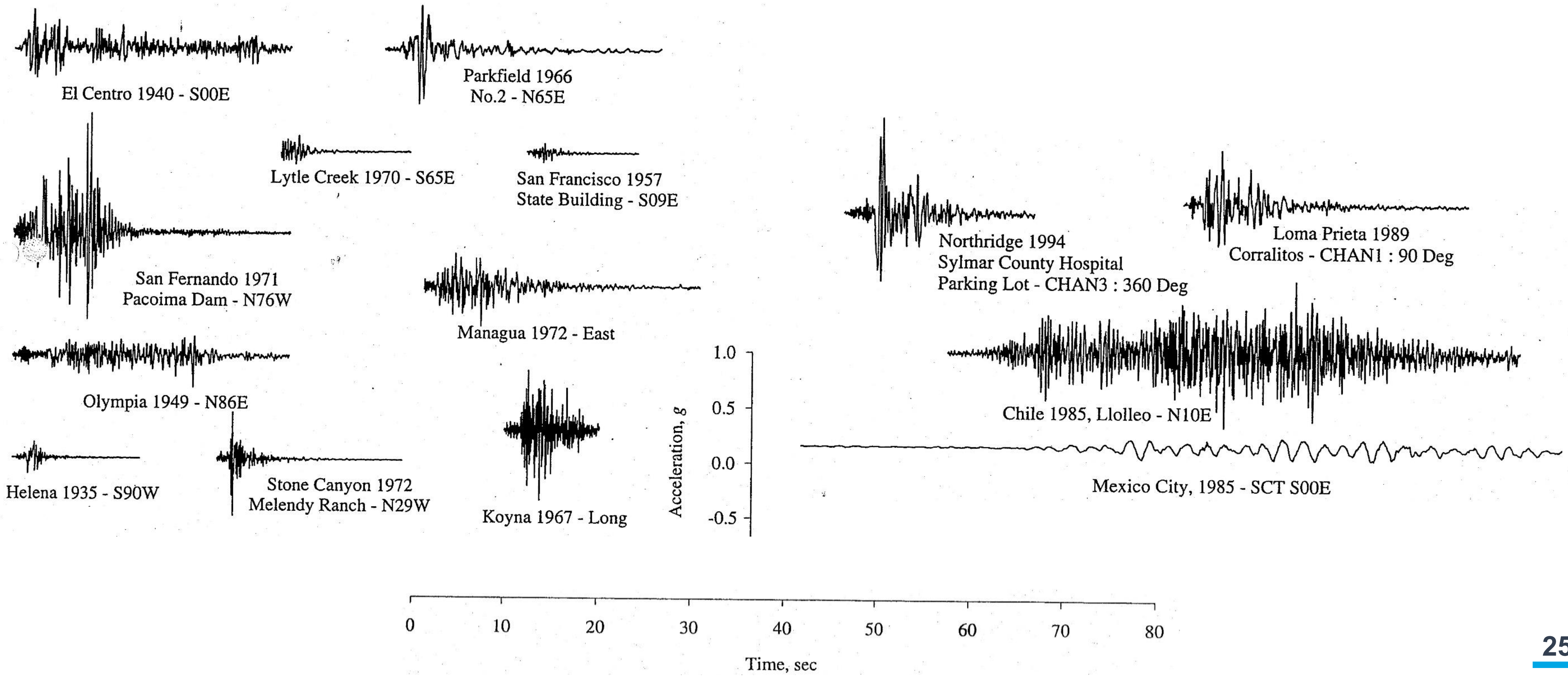
- Structural monitoring arrays
- Dense arrays, two and three dimensional
- Aftershock study arrays
- Local, regional and national seismic networks and arrays

# High Dynamic Range Strong Motion Accelerograph

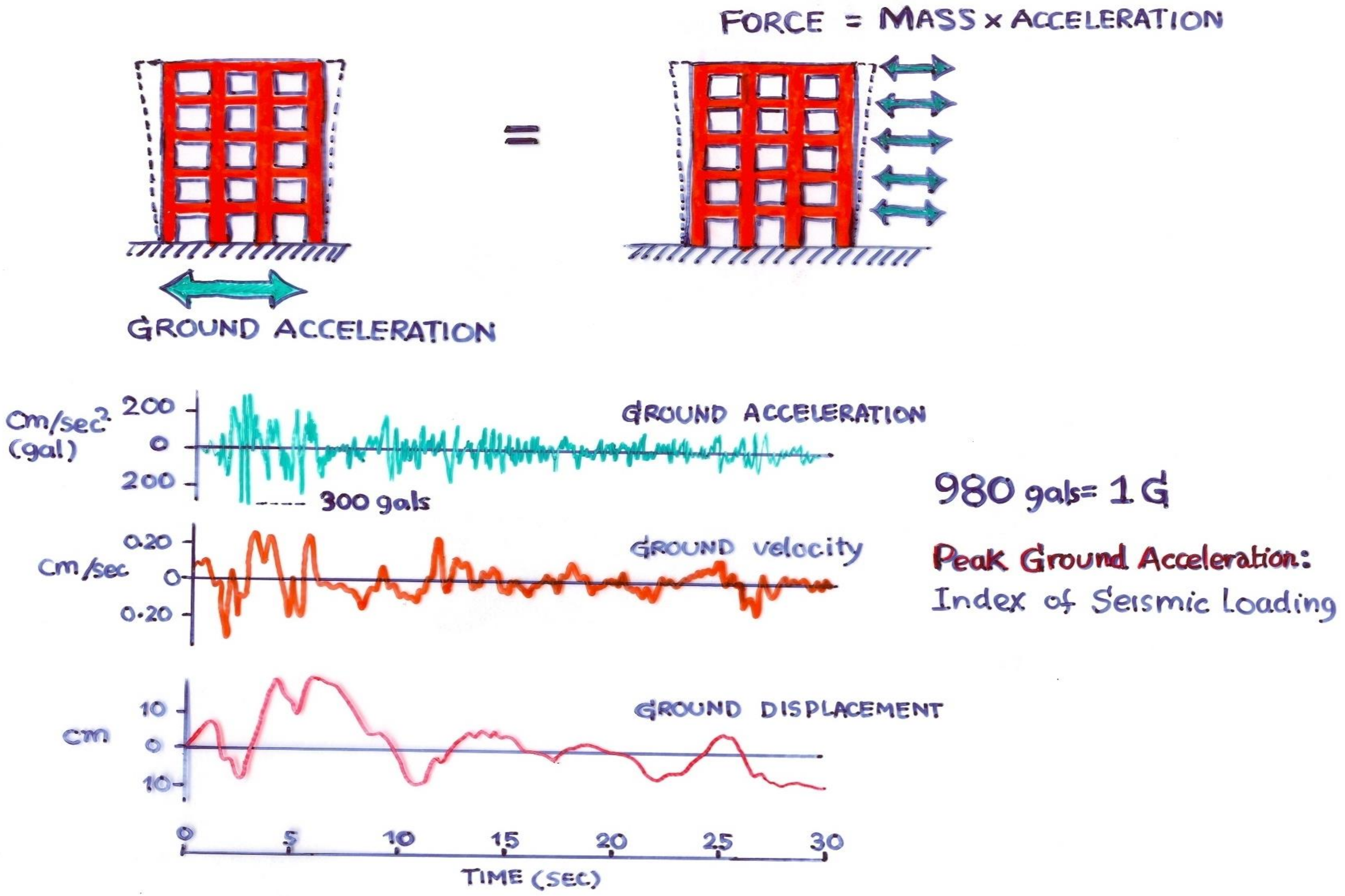




Source: M.D. Trifunac, M.I. Todorovska / Soil Dynamics and Earthquake Engineering 21 (2001), 275-286.

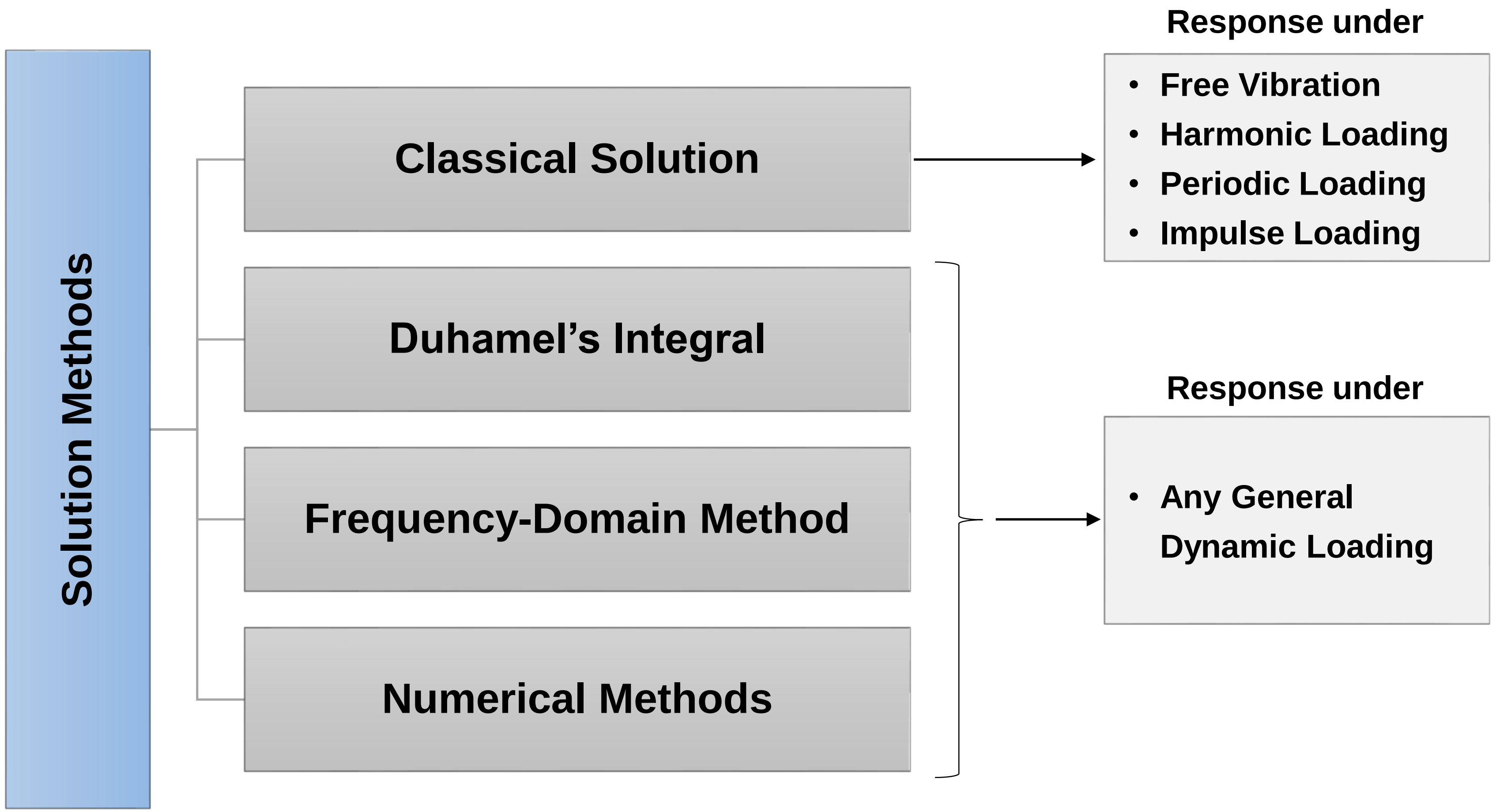
# Ground Acceleration Recordings



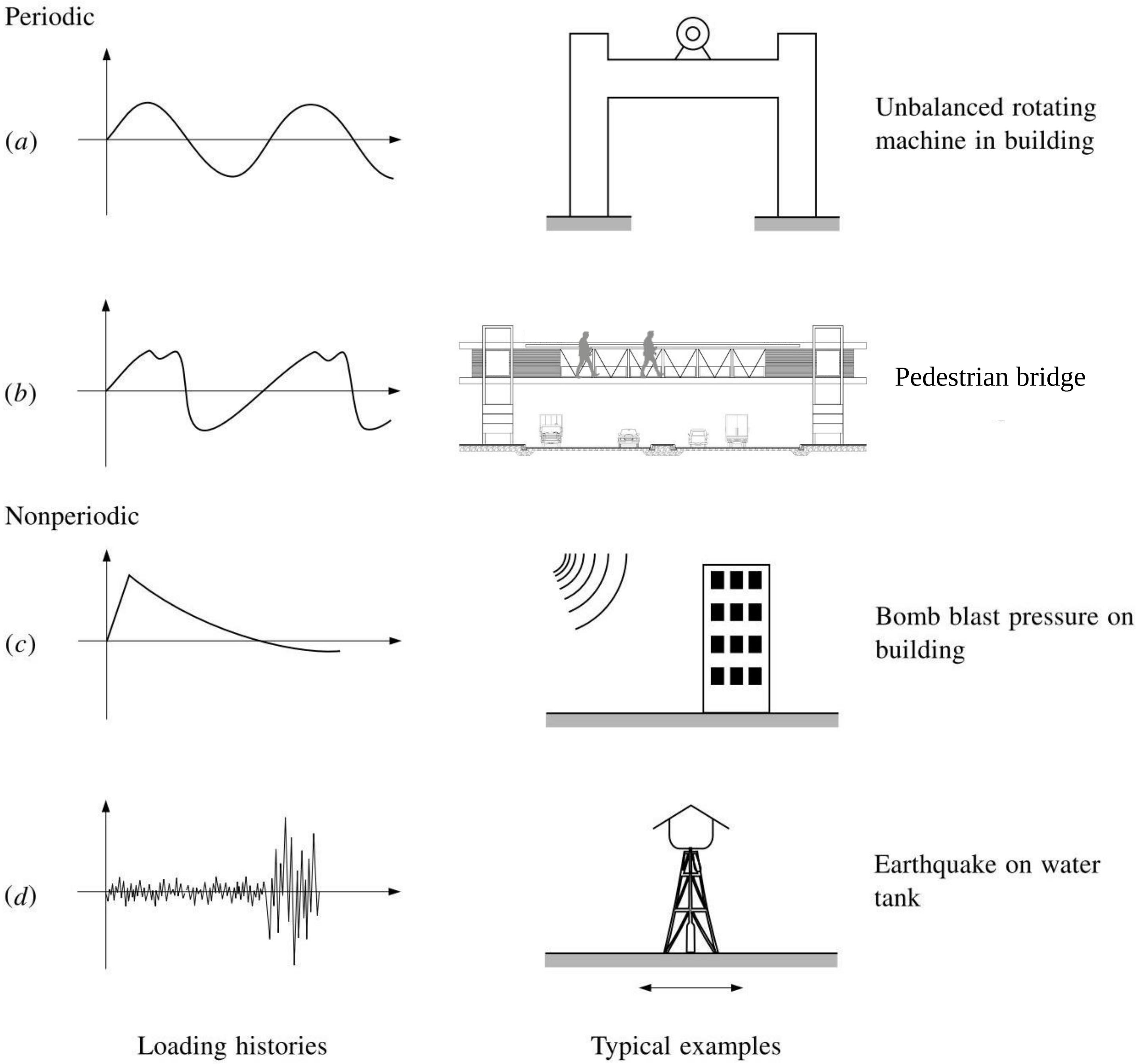
# Earthquake Loading on Structures



# Solution of Equation of Motion



# Types of Dynamic Loading on Structures




(a) Harmonic

(b) Periodic

(c) Impulsive

(d) Arbitrary

Modified from "Clough and Penzien (2003) Dynamics of Structures, 3rd Edition".

# Thank you