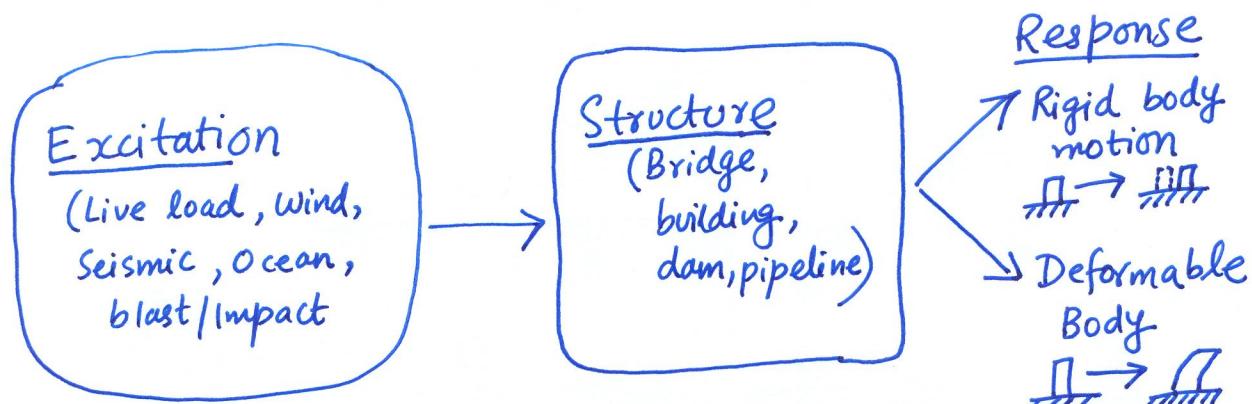


A Quick overview
of
Structural Dynamics

(Fawad A. Najam)

A Quick Overview of Structural Dynamics (Basic course)



Dynamic loading

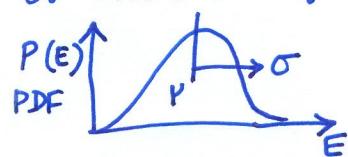
Deterministic

e.g. Rotating machine
or Last earthquake
(seismograph)



Probabilistic (Random)

e.g. Wind Load
or Next earthquake



Dynamic Loading

Periodic

Non-periodic

e.g. blast



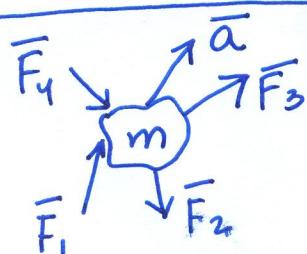
Often need to solve numerically

(e.g. harmonic)



Can Solve analytically

Dynamic Equilibrium (D'Alembert's Principle)



$$\sum \bar{F}_i = m \bar{a}$$

$-m\bar{a}$ = Inertial force

$$\sum \bar{F} = 0$$

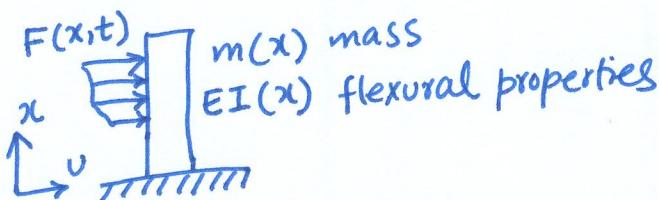
all forces including inertial force

Structural Models

(a mathematical representation
of a structure)

Continuous (Distributed Parameter)

- Realistic
- Very difficult to analyze



Governing equation:

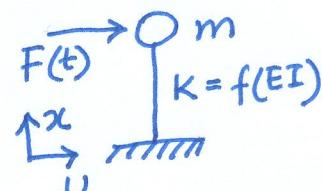
$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 x}{\partial x^2} \right) + m(x) \cdot \frac{\partial^2 u}{\partial t^2}$$

$$= F(x,t)$$

PDEs

Discrete

- Idealized
- Approximate
- Easy to analyze



Governing Equation:

$$m \frac{d^2 u}{dt^2} + Ku = F(t)$$

ODEs

- For "Linear Elastic" Discrete models, $K = \text{Constant}$ ($F = Ku \Rightarrow K = F/u$) 
- For "non-linear" Discrete models, K is a nonlinear function (a varying slope of $F-u$ relationship) 

Discrete Systems

Single-degree-of-freedom (SDF) Systems

Multiple-degree-of-freedom (MDF) Systems

Three ways of determining "Equations of Motions"

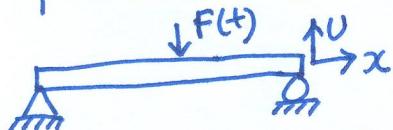
Dynamic Equilibrium

Virtual Work

Lagrange Equations
(Hamilton's Principle)

Methods of Discretization

Lumped mass procedure



Take time and position along span as two independent



now inertial forces can only developed at these three locations.

If only vertical motion (u) is allow — 3 DOF System
If rotation allowed — 6 DOF
" axial effect " — 9 DOF
If System is 3D — 18 DOF

Generalized displacement procedure

Assumption: The deflected shape of structure = Sum of series of specified displacement patterns

$$\text{so } U(x) = \dots$$

$$= \dots$$

$$+ \dots$$

$$+ \dots$$

$$U(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

In fact any arbitrary shape can be represented by an infinite series of assumed shapes (say $\Psi_n(x)$)

$$\text{so } U(x) = \sum Z_n \Psi_n(x)$$

amplitude

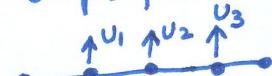
any assumed set of disp. fns

compatible with boundary conditions

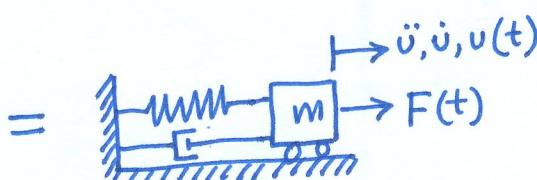
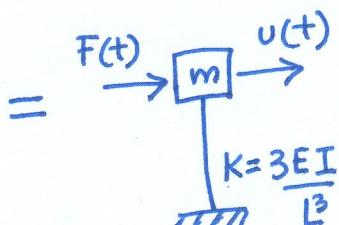
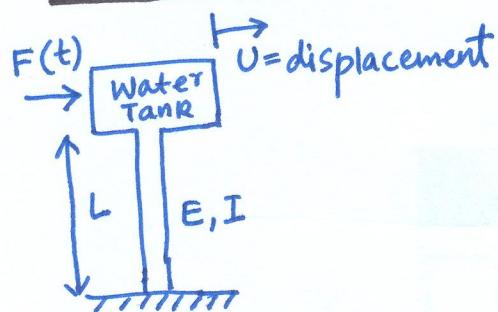
Finite Element Concept

(combination of first two approaches)

Z_n becomes nodal displacements and $\Psi_n(x)$ becomes Interpolation fns or shape fns.



SDF Systems



Damping force

Equilibrium equation of motion

$$m\ddot{U}(t) + C\dot{U}(t) + KU(t) = F(t)$$

(a second order ODE)

(Given)

$F(t)$
Excitation

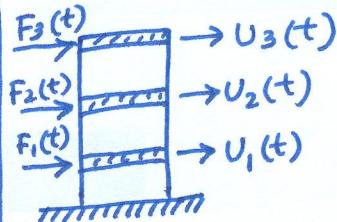
(can be determined)

M, K, C
Structure

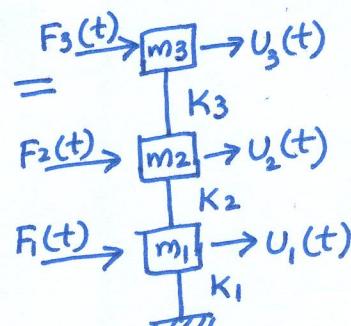
$U(t), \dot{U}(t), \ddot{U}(t)$
Response

(Basic Unknown)

MDF Systems



3 variables are required to describe structural dynamic motion



3 DOF system
3 eq. of motion
(1 for each story)

3 "Coupled" equations of motion : (without damping)

$$m_1\ddot{U}_1(t) + (K_1 + K_2)U_1(t) - K_2U_2 = F_1(t)$$

$$m_2\ddot{U}_2(t) + -K_2U_1(t) + (K_2 + K_3)U_2 - K_3U_3 = F_2(t)$$

$$m_3\ddot{U}_3(t) + K_3(U_3 - U_2) = F_3(t)$$

In matrix form,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{U}_1(t) \\ \ddot{U}_2(t) \\ \ddot{U}_3(t) \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 & 0 \\ -K_2 & K_2 + K_3 & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix}$$

$$\text{or } M\ddot{U} + KU = F(t)$$

Including damping,

$$M\ddot{U}(t) + (C\dot{U}(t) + KU(t)) = F(t)$$

The coupling terms in these equations represent the interaction b/w dynamic equilibrium at different stories.

Dynamics of SDF Systems

$$m\ddot{u}(t) + c\dot{u}(t) + Ku(t) = f(t)$$

Lets define $\xi = \frac{c}{2\sqrt{mk}} = \text{critical damping ratio}$

circular natural frequency $= \omega$

$$\omega = 2\pi f$$

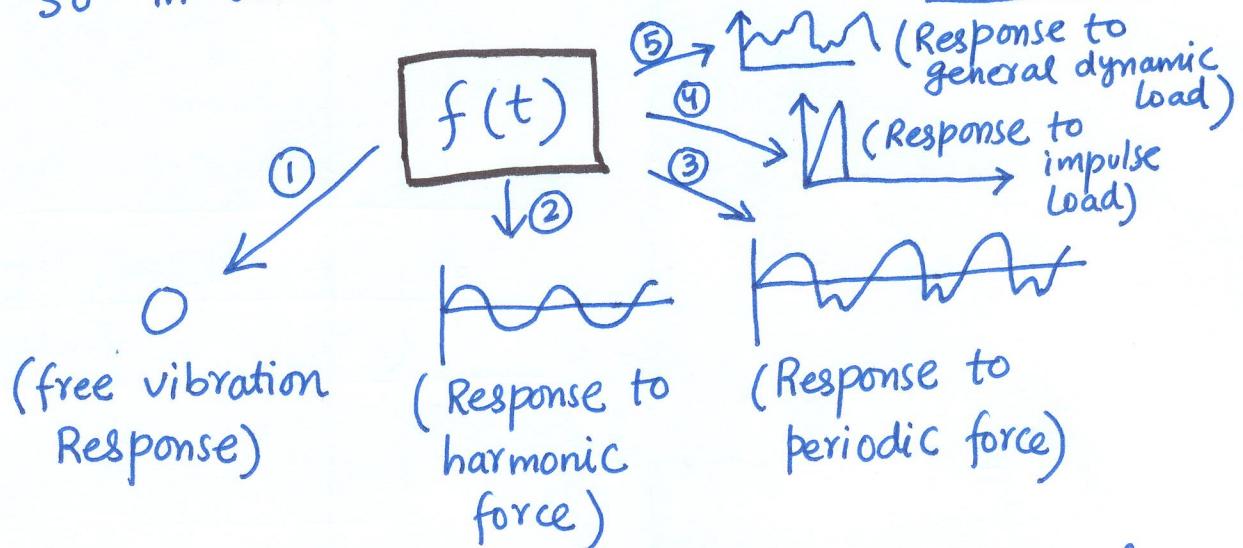
$$f = \frac{1}{T}$$

$$T = 2\pi \sqrt{\frac{m}{k}}$$

so $\omega = \sqrt{k/m}$ natural frequency
of SDF system is
only function of m, k

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = f(t)$$

So in order to solve, you need ξ and ω



Response to earthquakes \Rightarrow ⑤ Response to general dynamic load

- Assumes Linear system
- Closed form solution not always possible e.g. (earthquakes)

Duhamel's Integral (convolution)

Numerical Integration (step-by-step)

Can accommodate nonlinear systems \leftarrow All seismic analysis software

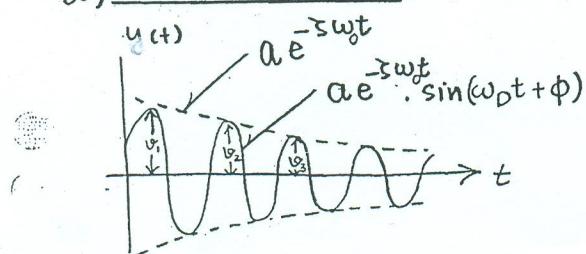
For detailed solution of all five cases

①, ②, ③, ④, ⑤ Duhamel Integral
Numerical Integration

Please refer to detailed "Structural Dynamics" notes.

A Quick Summary

a) Free Vibration: $f(t) = 0$



a and ϕ depend on initial displacement and velocity.

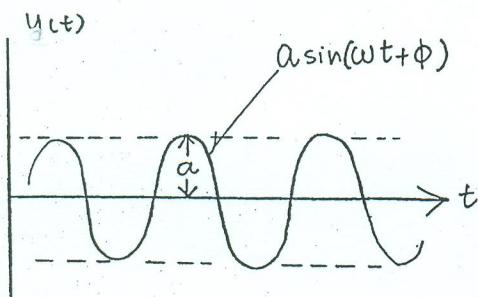
$$\omega_D = \omega_0 \sqrt{1 - \zeta^2} \approx \omega_0 \quad (\zeta \ll 1)$$

Logarithmic decrement: δ

$$\delta \equiv \ln \left(\frac{\vartheta_n}{\vartheta_{n+1}} \right) \approx 2\pi\zeta$$

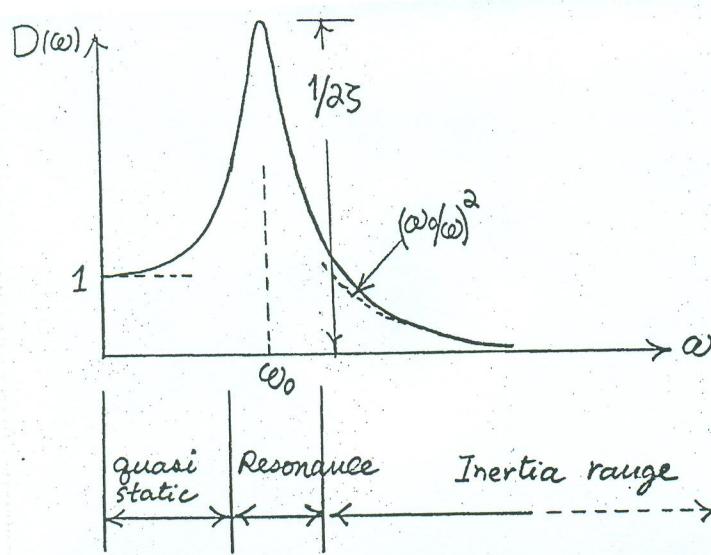
Critical damping ratio determines the rate of amplitude decay (or the dissipate rate of vibration energy)

b) Steady-state response to harmonic force: $f(t) = P \sin \omega t$

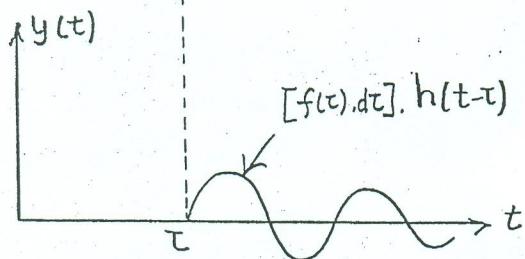
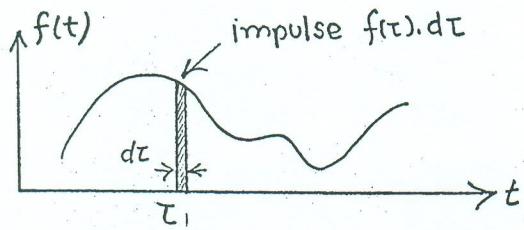


$$a = \underbrace{\left(\frac{P}{K} \right)}_{\text{Static displacement}} \cdot \underbrace{D(\omega)}_{\text{Dynamic Magnification factor}}$$

$$D(\omega) = \frac{1}{\sqrt{(1 - \omega/\omega_0)^2 + (2\zeta\omega/\omega_0)^2}}$$



C. Response to arbitrary forces



an arbitrary force $f(t)$ = a series of impulses

$h(t)$ = unit impulse response.

(response to a unit impulse acting on the structure at time $t=0$)

$$h(t) = \begin{cases} \frac{1}{m\omega_0} e^{-j\omega_0 t} \sin \omega_0 t & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Response to the impulse $f(\tau) \cdot d\tau$ which is acting at time $t=\tau$ is:

$$[f(\tau) \cdot d\tau] \cdot h(t-\tau)$$

Response to the arbitrary force $f(t)$ is the superposition of all impulse responses:

$$y(t) = \int_{-\infty}^t f(\tau) \cdot h(t-\tau) d\tau$$

Duhamel Integral
or Convolution Integral.

(loading starts from $t=-\infty$)

(The integral valids only in the case of linear structures)

P

Dynamics of MDF Systems

Lets consider the most simple case : Free vibration
Response + Undamped

$$IM\ddot{U}(t) + IKU(t) = \Phi$$

Free-vibration response of a SDF system, $U(t) = \underline{a} \sin(\omega t + \theta)$
 ω = natural circular frequency.

So by analogy,

$$U(t) = \underline{\phi} \sin(\omega t + \theta)$$

an N-vector that
represents the shape
of vibration

arbitrary
amplitude as per
the initial conditions.

Putting in above equation and Solving,

$$IK\underline{\phi} = \omega^2 IM\underline{\phi}$$

$$[IK - \omega^2 IM]\underline{\phi} = \Phi$$

Eigen-value Eigen-vector

For $|A|\mathbf{X} = \Phi$, The equation has non-zero solution
of \mathbf{X} if and only if $\text{Det } A = 0$ (Cramer's Theorem)

$$\text{so } \text{Det}(IK - \omega^2 IM) = 0$$

This will yield "frequency equation" — an
nth degree polynomial ($n = \text{DOFs}$) with "n"
roots $\omega_1, \omega_2, \omega_3, \dots, \omega_n$ — (frequencies of n
modes of vibration)

For each $\omega_i \rightarrow$ a corresponding $\underline{\phi}_i$ can be calculated
using above equation.

$\underline{\phi}_i$ is a "free vibration mode shape" of an
ith mode of vibration.

Since $[K]$ and M are $n \times n$ matrices

so $\text{Det} [IK - \omega^2 M] = 0$ will give a polynomial of degree n . \rightarrow It's called "frequency equation".

The n roots are $\omega_1^2, \omega_2^2, \omega_3^2, \dots, \omega_n^2$.

For each ω , there is a corresponding ϕ vector which makes $[IK - \omega^2 M]\phi = 0$

If ϕ_i is a solution, then $a\phi_i$ is also a solution ($a = \text{arbitrary constant}$). So we can better think of it as a "Shape".

So
There are many ϕ vectors (vector of amplitudes, vibration shape vector) possible, each corresponding to an ω value, which make $[K]\phi - \omega^2 M\phi = 0$.

so ϕ_i is a mode shape (corresponding to ω_i).

If we deform the structure statically into a mode shape ϕ_i then set it free, it will oscillate b/w the initial deformed shape and the negative of the initial deformed shape at a frequency ω_i .

The total displacement = sum of each mode at its natural frequency.

Since this eigen-value problem only has real values in all matrices, the displacement vector of mode shapes is real. It implies that the motion of all locations in the FE model is either perfectly in phase (0°)

relative phase or (180°) out-of-phase. perfectly.
relative phase



In complex mode shape, Every location can have a different phase angle (looks like a wave when we animate it)

So What a mode is :-

A combination of a deformed shape in which a structure will exchange kinetic and strain energies continuously. The frequency at which it occurs is natural frequency.

It is the property of structure since it is calculated without any load applied to the structure.

Magnitude of a mode shape is arbitrary \rightarrow so mode shape is not a displaced shape. But since both mode shape and displacement shape vector "imply" displacement of ^{actual} structure, it is entirely (esp. mode shape) possible for a software to determine stresses, strains etc. etc. for each mode shape. (But these values are almost useless)

But their distribution is useful, although values are useless.

Think of mode shape as a displacement with random scale factor applied. The stress for a mode shape would be the stress you would

Obtain if the actual displacement of the structure is exactly the same as that of mode shape "with arbitrary values".

e.g. if software gives max stress of 25 GPa for first mode and at that point it gives the value of first mode shape (amplitude) = 716 mm.

It means if actual structure is only responding in first mode, and the stress value for 716 mm disp is 25 GPa. These values have nothing to do with actual displacement or stresses. Let's say the actual first mode displacement of structure is

$$10 \text{ mm} \quad \text{then the Peak stress} = \frac{25 \text{ GPa}}{716 \text{ mm}} \times 10 \text{ mm}$$
$$= 349.1 \text{ MPa}$$

So

- A mode \neq displacement, mode = Shape
- It is not a structural response due to any input loading.
- The results of modal analysis (disps, stresses etc) not representative of any dynamic loading. Only the relative values b/w any two points can have any meaning, not the numerical values.

The purpose of modal analysis is to find the shapes and frequencies at which the

structure will amplify the effect of a load.

What modal analysis used for :-

a) Finding loose components :

If the structure is under-constrained, analysis will result a 0-hz (i-e static) mode for each unconstrained direction. They will not be exactly 0-hz for two reasons i-e round off or eigen-value analysis is an iterative process. But f will be very low, like 0.001 hz. These modes are also referred to as "rigid-body modes" or "strain-less modes".

This is because the structure (or part of structure) translates or rotates in such a way that no stress results, i-e moves in some direction as it was rigid.

So the displacement shape of these modes provide enough information about which component may be loose or which constraints are missing.

b) Deciding which rotational speeds are dangerous. ^{rotor dynamics}

c) Where to constrain or load a structure?

The role of nodes etc.

d) Finding out how to move a mode $T = 2\pi\sqrt{\frac{m}{K}}$.

Undamped free vibration Response of MDF Systems

So the general solution of free vibration of mdof system is

$$v(t) = a_1 \phi_1 \sin(\omega_1 t + \theta_1) + a_2 \phi_2 \sin(\omega_2 t + \theta_2) \\ + \dots + a_n \phi_n \sin(\omega_n t + \theta_n)$$

a_1, a_2, a_3 are six arbitrary constants
 $\theta_1, \theta_2, \theta_3$ to be determined from initial conditions $v(0), \dot{v}(0)$

mode shape

$$v(t) = \sum_{i=1}^N a_i \phi_i \sin(\omega_i t + \theta_i)$$

(Time varying amplitude)

so we formulate IK and IM then software solves $IK\phi = \lambda M\phi$ and gives

$$\{\lambda_1, \lambda_2, \lambda_3, \dots\} \text{ and } \{\phi_1, \phi_2, \phi_3, \dots\}$$

so finally $\omega_i = \sqrt{\lambda_i}$ and ϕ_i 's.

We can assemble N modes into a

"Modal Matrix"

$$\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_{1N} \\ \phi_2 & & \vdots \\ \phi_{31} & & \vdots \\ \vdots & & \vdots \\ \phi_{N1} & & \phi_{NN} \end{bmatrix}$$

Spectral Matrix = $\Omega^2 = \begin{bmatrix} \omega_1^2 & & \\ & \omega_2^2 & \\ & & \ddots & \ddots & \omega_n^2 \end{bmatrix}$
 diagonal matrix

so

$$K \phi_n = M \phi_n \omega_n^2$$

scalars

so using modal and spectral matrices, we can assemble all relationships in this form,

$$K\Phi = M\Phi\Omega^2$$

Orthogonality of Modes

If $V^T A W = 0$

we say that V is orthogonal to W with respect to vector A .

We will show that

"Mode shape matrix is orthogonal to it self w.r.t matrix M and K ." for each

$i \neq j$

$$K \phi_i = \omega_i^2 M \phi_i \quad \text{--- (A)}$$

$$K \phi_j = \omega_j^2 M \phi_j \quad \text{--- (B)}$$

$$\phi_j^T \times \text{eq (A)} \quad \text{--- (C)}$$

$$\phi_i^T \times \text{eq (B)} \quad \text{--- (D)}$$

$$[\text{(C)}]^T \Rightarrow \phi_i^T K^T \phi_j = \omega_i^2 \phi_i^T M^T \phi_j$$

$$\phi_i^T K \phi_j = \omega_i^2 \phi_i^T M \phi_j \quad \text{--- (E)}$$

K and M are symmetric

scalar = 0

$$(w_j^2 - w_i^2) \cdot \phi_i^T M \phi_j = 0$$

i.e. each column is a modal vector,
According to orthogonality principle,

$$\Phi^T M \Phi = \begin{bmatrix} \phi_1^T \\ \vdots \\ \phi_N^T \end{bmatrix} \cdot M \begin{bmatrix} \phi_1 & \cdots & \phi_N \end{bmatrix}$$

e.g. for a 10-story building with N-modes (<sup>1 value
of ϕ_i at
each story</sup>)

$$\Phi^T = N \times 10$$

$$\Phi^T M \Phi = N \times N \text{ matrix}$$

$$\Phi^T M \Phi = \begin{bmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{bmatrix}$$

Similarly

$$\Phi^T K \Phi = \begin{bmatrix} \mu_1 \omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \omega_n^2 \end{bmatrix}$$

The value of μ_i depends on how we scale
 ϕ_i . If we normalize mode shapes such that
 $\phi_i^T M \phi_i = 1$ then they are called orthonormal
relative to mass matrix.

In practice usually they are normalized such that
their top most ordinate = 1. e.g.



It means when $i \neq j$ (i.e. $\omega_i^2 \neq \omega_j^2$)

$$\phi_i^T M \phi_j = 0$$

Similarly $\phi_i^T K \phi_j = 0$

so mode shapes ϕ_i and ϕ_j are orthogonal to each other w.r.t M and K (as long as $i \neq j$).

when $i=j$, $\phi_i^T K \phi_i$ or $\phi_i^T M \phi_i$ will have some positive number ($i=j$ means diagonal)

so let's call that diagonal entries in $\phi_i^T M \phi_i$.

Square matrix = Modal mass so

$$\phi_i^T M \phi_i = \mu_i \text{ (scalar } > 0\text{)}$$

and since we have

$$\phi_i^T K \phi_j = \omega_j^2 \phi_i^T M \phi_j$$

so $\phi_i^T K \phi_i = \omega_i^2 \mu_i$

so

Summary

$$\boxed{\begin{aligned} \phi_i^T M \phi_j &= \begin{cases} 0 & \text{for } i \neq j \\ \mu_i & \text{for } i=j \end{cases} \\ \phi_i^T K \phi_j &= \begin{cases} 0 & \text{for } i \neq j \\ \mu_i \omega_i^2 & \text{for } i=j \end{cases} \end{aligned}}$$

so if we define a modal matrix

$$\text{Modal Matrix} = \Phi = [\phi_1 \ \phi_2 \ \phi_3 \ \dots \ \phi_N]$$

(5)

Interpretation of modal Orthogonality :-

(a) Work done by the n -th mode inertia forces in going through the r -th mode displacements

$$= 0$$

(b) Work done by equivalent static forces associated with displacements in n th mode in going through the r -th mode displacements

$$= 0$$

For proof \rightarrow Chopra section 10.5.

Modal Expansion of displacements :-

Any set of N independent vectors can be used as a basis for representing any other vector of order N . So if we use modeshapes as such a basis,

$$U = \sum_{r=1}^N \phi_r q_r = \Phi q$$

q_r are scalar multipliers called "modal coordinates" or "normal coordinates".

$$\phi_n^T M U = \sum_{r=1}^N \phi_n^T M \phi_r q_r$$

Because of orthogonality, $\phi_n^T M \phi_r = 0$ except $r=n$

$$\text{so } \phi_n^T M U = \phi_n^T M \phi_n q_n$$

$$q_n = \frac{\phi_n^T M U}{\phi_n^T M \phi_n} = \frac{\phi_n^T M U}{M_n}$$

↙
modal mass

Modal Analysis

The concept of generalized Coordinates :-

The displacement of a NDOF system $\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix}$

can also be defined by a different set of coordinates. e.g. $\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{Bmatrix}$ where the relation b/w

qV and U is linear. i.e

$$U = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NN} \end{Bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{Bmatrix} + \dots$$

or

$$U = \underbrace{\phi_1 \cdot q_1(t) + \phi_2 q_2(t) + \dots + \phi_N q_N(t)}_{\text{A vector defining any vibration shape (dimensionless + constant)}}$$

A time-varying function (Amplitude)

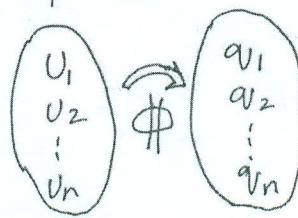
or

$$U = \underbrace{\phi qV(t)}_{\substack{\text{A vector of displacements} \\ \text{Transformation Matrix}}} \quad \underbrace{\phi}_{\text{A vector of generalized coordinates}}$$

For one-to-one linear transformation, ϕ should be a set of independent vectors

$$\phi_1, \phi_2, \dots, \phi_N.$$

(then inverse transformation is possible $qV = \phi^{-1}U$)



Classical Modal Analysis

To determine the "Forced" Vibration Response of MDF Systems :-

The underlying idea is

"If we use Φ matrix as transformation matrix to convert displacement $U(t)$ into generalized coordinates $q(t)$, the equations of motion (in $q(t)$ coordinates) will be uncoupled".

So we can solve them separately, each equation will be like the equation of an SDF system.
So we can decompose MDF in to sum of SDF systems.

This uncoupling (in $q(t)$ coordinates) is due to the orthogonality property of mode shapes.

Equations of Motion in Normal Coordinates:-

"Uncoupled form of equations"

We apply modal expansion of displacement.

$$M\ddot{U} + KU = Q(t)$$

$$U(t) = \sum_{i=1}^N \phi_i q_i(t) \quad (\text{Transformation b/w system of } U \text{ coordinate to } \phi \text{ coordinate})$$

in matrix form,

$$U(t) = \Phi \mathbf{q}(t)$$

$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{Bmatrix} \quad (\text{a vector of new generalized coordinates})$$

Φ is used as transformation matrix - $N \times N$

Consists of N independent, orthogonal vectors.

$$\mathbf{q}(t) = \Phi^{-1} U(t)$$

so introducing this expansion,

$$M\ddot{\Phi}\mathbf{q}(t) + K\Phi\mathbf{q}(t) = Q(t)$$

multiply by Φ^T

$$\Phi^T M \ddot{\Phi} \mathbf{q}(t) + \Phi^T K \Phi \mathbf{q}(t) = \Phi^T Q(t)$$

Applying orthogonality conditions,

$$\begin{bmatrix} M & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_N \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_N \end{bmatrix} + \begin{bmatrix} M\omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_N\omega_N^2 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} = \begin{bmatrix} R_1(t) \\ \vdots \\ R_N(t) \end{bmatrix}$$

Where $R(t) = \begin{bmatrix} R_1(t) \\ \vdots \\ R_N(t) \end{bmatrix} = \Phi^T Q(t)$

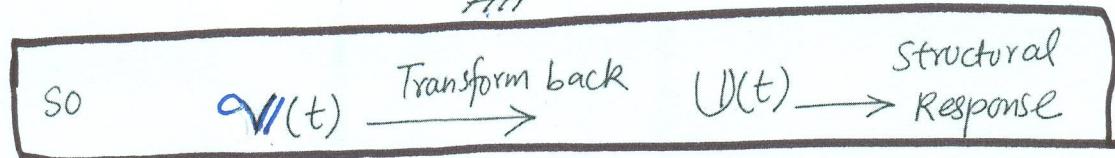
Now the equations are uncoupled. Each equation can be written as

$$M_i \ddot{q}_i(t) + K_i q_i^2(t) = R_i(t)$$

$$R_i(t) = \phi_{1i} Q_1(t) + \phi_{2i} Q_2(t) + \dots + \phi_{ni} Q_n(t)$$

Each equation is like equation of a SDOF

$$\begin{array}{c} F(t) = R_i(t) \\ m = M_i \\ K = K_i w_i^2 \end{array}$$



Modal Analysis :-

- a) Strictly speaking, applicable to linear systems
- b) Simplify problem to a reduced no. of modes.
- c) Not applicable for non-proportional damping.

$$\Phi^T C \Phi = \text{Non-diagonal Matrix}$$

In this case, complex modal analysis is needed

$$\left. \begin{array}{l} \text{If } C = \alpha M \text{ or} \\ C = \beta K \text{ or} \\ C = \alpha M + \beta K \end{array} \right\} \rightarrow \begin{array}{l} \text{equations are uncoupled} \\ (\text{Proportional damping}) \end{array}$$

For non-proportional, Response from 1 mode will affect other, but usually F_D is very low.

- d) Not applicable to Non-linear structures.

In practice, it is convenient to include the effect of damping in terms of "Modal damping".

for i th mode, $M_i \ddot{\gamma}_i(t) + 2\xi_i \omega_i \dot{\gamma}_i + \kappa_i \omega_i^2 \gamma_i = R_i(t)$

$\underbrace{2\xi_i \omega_i \dot{\gamma}_i}_{\text{additional damping term.}}$

ξ_i represents a "Lumped" quantity of energy dissipation.

In SDOFs, we can use tangent K at any instant and tangent $C \rightarrow$ we update these quantities. In MDOFs it is not possible to do this. So

$$D = \begin{bmatrix} 2\xi_1 \omega_1 \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -2\xi_N \omega_N \mu_N \end{bmatrix}$$

Where $\Phi^T C \Phi = D$

$$C = (\Phi^T)^{-1} D \Phi^{-1}$$

It can be shown that

$$\Phi^{-1} = [M]^{-1} \Phi^T M$$

$$[\Phi^T]^{-1} = M \Phi [M]^{-1}$$

so $C = M \Phi [M]^{-1} D [M]^{-1} \Phi^T M$
 $= M \Phi \tilde{D} \Phi^T M$

Where

$$\tilde{D} = \begin{bmatrix} 2\xi_1 \omega_1 / \mu_1 & 0 \\ \vdots & \ddots \\ 0 & 2\xi_N \omega_N / \mu_N \end{bmatrix}$$

Modal Analysis Summary :-

The dynamic response of a MDF system to external forces $P(t)$ can be computed by modal analysis.

① Define structural properties

- a) IM, IK,
- b) estimate ξ

② Determine ω_n and modes ϕ_n

③ Compute response for each mode

$$\text{solve } M_n \ddot{q}_n(t) + C_n \dot{q}_n(t) + K_n q_n(t) = P_n(t)$$

$$\text{a) or } \ddot{q}_n + 2\xi_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n}$$

for $q_n(t)$

b) Determine nodal displacements using $U_n(t) = \phi_n \times q_n(t)$

c) Determine element forces associated by nodal displacements $U_n(t) \rightarrow \gamma_n(t)$

(i) Using element stiffness properties

(ii) Using static analysis at each time step, under equivalent static forces $f_n = K U_n(t)$

$$f_n = \omega_n^2 m \phi_n q_n(t)$$

④ Combine all modal contributions.

$$U(t) = \sum_{n=1}^N U_n(t) = \sum_{n=1}^N \phi_n q_n(t)$$

$$\gamma(t) = \sum_{n=1}^N \gamma_n(t)$$

Classical Modal Analysis

(a) Equations of motion in physical coordinates:-

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

3 coupled Ordinary diff. eqs.

$$m(s)\frac{\partial^2 x}{\partial t^2} + c(s)\frac{\partial x}{\partial t} + \frac{\partial^2}{\partial s^2}(EI(s)\frac{\partial^2 x}{\partial x^2}) = F(s,t)$$

Partial diff. Eq.

(b) Linear transformation :-

$$\begin{aligned} x_1(t) &= \\ x_2(t) &= \\ x_3(t) &= \\ \underline{x}(t) &= \underline{\Phi}_1 \cdot \underline{y}_1(t) + \underline{\Phi}_2 \cdot \underline{y}_2(t) + \underline{\Phi}_3 \cdot \underline{y}_3(t) = \underline{\Phi} \underline{y}(t) \end{aligned}$$

displacement vector in physical coordinates

model vector

model Amplitude

(c) Equations of motions in modal coordinates:-

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} + \begin{bmatrix} R_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

3 uncoupled ordin. diff. eqs.

∞ uncoupled ordin. diff. eqs.

$$m_i \ddot{y}_i + c_i \dot{y}_i + k_i y_i = f_i(t)$$

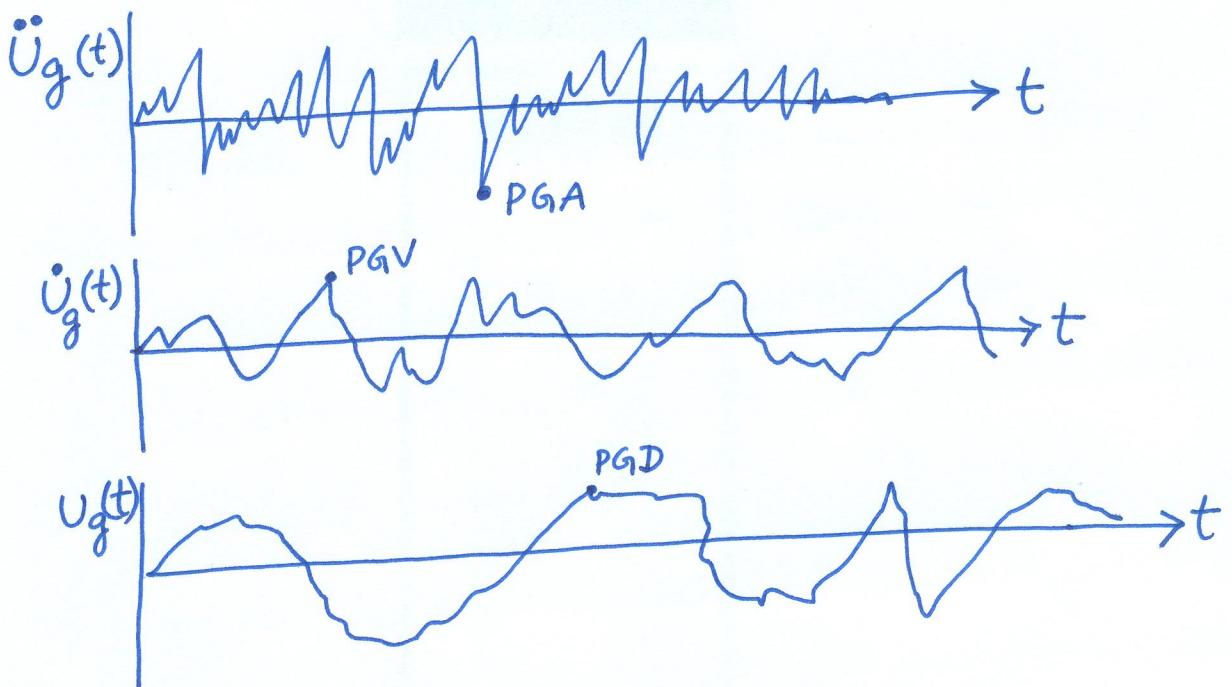
$$i = 1, 2, \dots, \infty$$

$$\begin{aligned} m_i &= \int_0^L m(s) \phi_i^2 s ds ; \quad c_i = \int_0^L c(s) \phi_i^2 s ds \\ k_i &= \int_0^L EI(s) (\frac{\partial^2 \phi_i}{\partial s^2})^2 ds ; \quad f_i(t) = \int_0^L F(s,t) \phi_i(s) ds \end{aligned}$$

* Regardless to structural complexity, we can always break the system down to a set of "equivalent SDOF systems".

\Rightarrow Necessity to understand the dynamic behaviours of SDOF systems

Response of SDOF systems under Earthquake ground Motions



$$P_{eff}(t) = -m\ddot{u}_g(t)$$

equation of motion:-

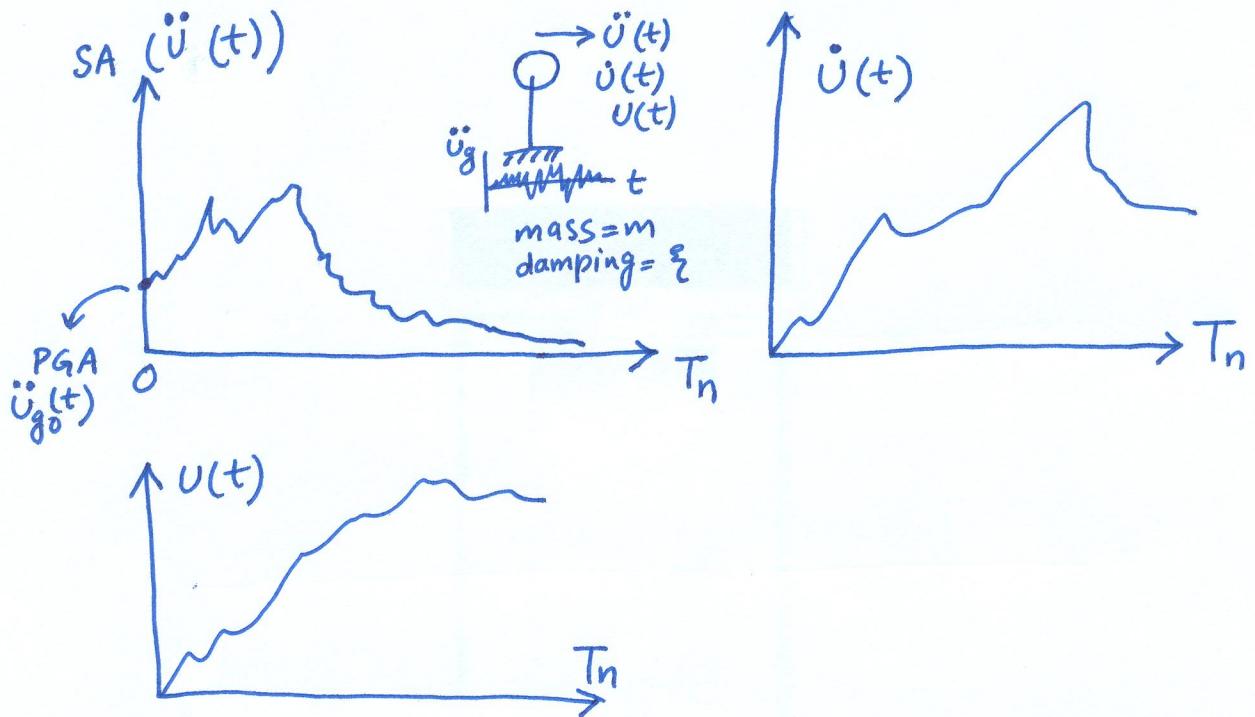
$$m\ddot{u}(t) + c\dot{u}(t) + K u(t) = -m\ddot{u}_g(t)$$

$$\ddot{u}(t) + 2\xi\omega_n\dot{u}(t) + \omega_n^2 u(t) = -m\ddot{u}_g(t)$$

so for a given ground motion, the response depends only on ω_n (or T_n) and ξ of SDF system.

Response spectrum :- A plot of the peak values of a response quantity of SDF system ($u(t)$, $\dot{u}(t)$ or $\ddot{u}(t)$) as a function of $T_n \Rightarrow$ RS of that quantity.

(It summarize the peak response of all possible linear SDFs to a particular GM)

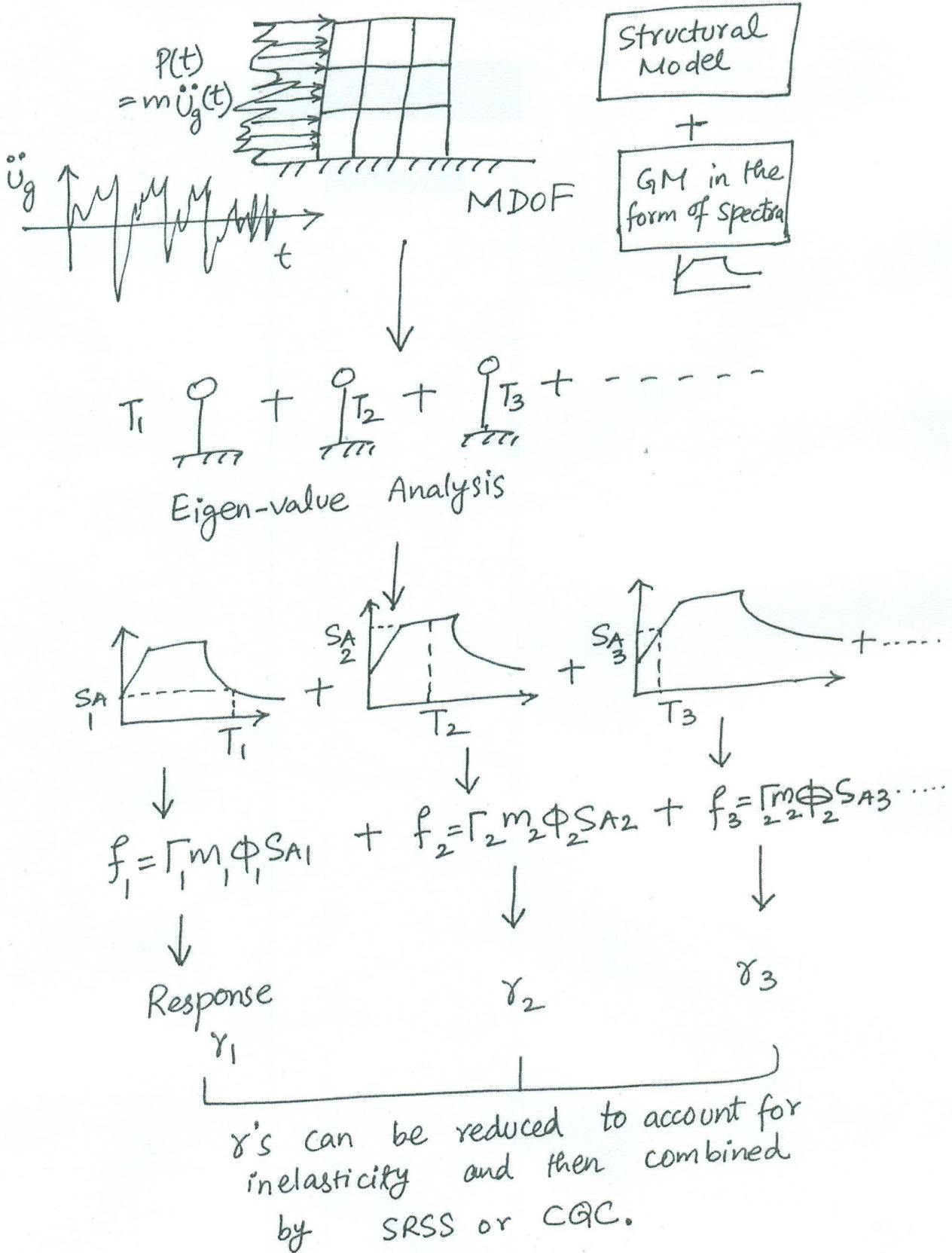


RS provides a practical approach to apply the knowledge of structural dynamics to the design of structures and development of lateral force Requirements in building codes.



Response spectrum Analysis procedure

Response spectrum Method

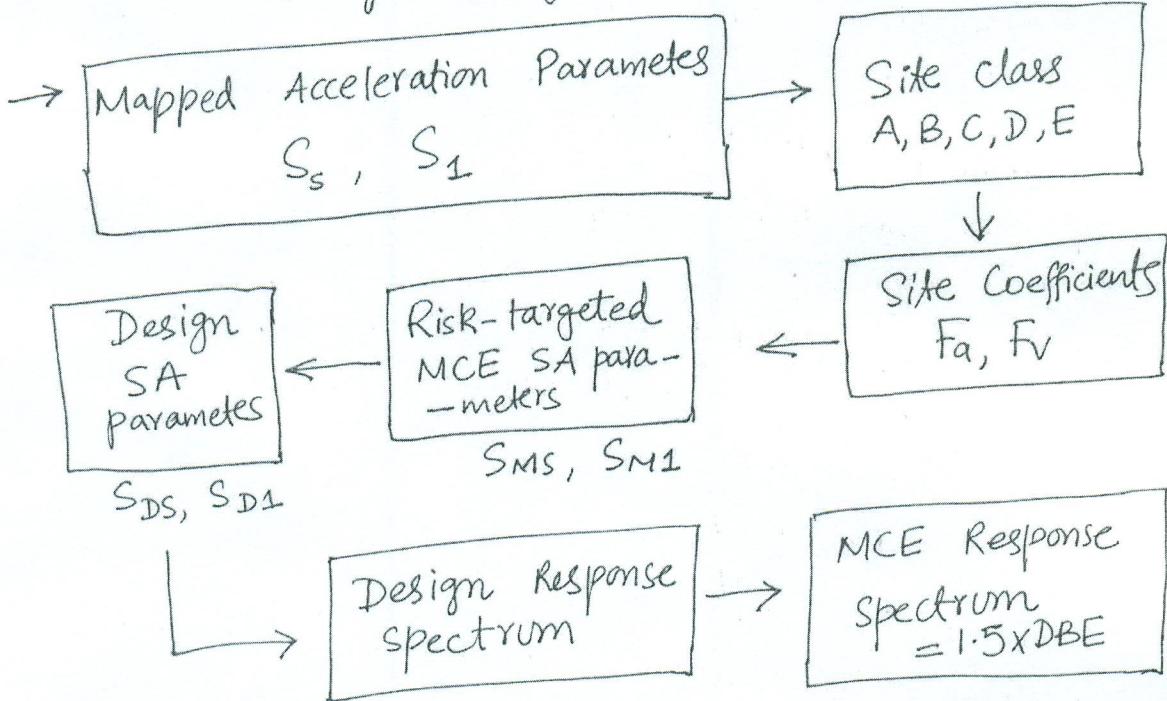


Code-based Seismic Design

→ Seismic Design Category (SDC) → assigned to a structure based on its risk category, and the severity of design EQ motion.

→ Nominal strength → without any reduction factor.

$$\text{Design strength} = \text{Nominal strength} \times \phi$$



→ Table 1.5-1 → Risk Category of Buildings I, II, III, IV

→ Table 1.5-2 → Importance Factors of Risk Category of buildings. $I_e = 1$ (for I and II)
 $= 1.25$ for III
 $= 1.5$ for IV.

→ SDC → based on S_{DS} or S_{D1} and Risk Category

$$A, B, C, D, E, F$$

$\downarrow \quad \uparrow$

$S_1 > 0.7S_g \quad S_1 > 0.7S_g$
 for I, II, III for IV

(1)

→ For SDC D, E, F → PGA shall be determined based on (a) A site-specific study taking into account soil amplification effects or (b) $PGA_M = F_{PGA} PGA$

adjusted for site class effects ↓
site coefficient Table 11.8-1

→ R, Ω_0 and C_d (Table 12.2-1)

↓ ↓ ↓
base shear element design forces design story drift

→ Combining factored loads using strength design.
Basic combinations

- 1) $1.4D$
- 2) $1.2D + 1.6L + 0.5(L_y, S \text{ or } R)$
- 3) $1.2D + 1.6(L_y, S \text{ or } R) + (L \text{ or } 0.5W)$
- 4) $1.2D + 1.0W + L + 0.5(L_y, S \text{ or } R)$
- 5) $1.2D + 1.0E + L + 0.2S$
- 6) $0.9D + 1.0W$
- 7) $0.9D + 1.0E$

→ Redundancy → sometimes synonym of "Alternative loading path". The ability of structure to redistribute among its members/connections the loads which can no longer be carried by some other damaged portions. Non-redundant structures → fail immediately under local damage.

A " ρ " redundancy factor is assigned to seismic force resisting system.

$$\rho = 1 \quad (\text{SDC B, C and } \dots)$$

$$\rho = 1.3 \quad (\text{SDC D, E, F except } \dots)$$

→ The "E" in load combination 5 is

$$E = E_h + E_v$$

For load combination 7, $E = E_h - E_v$

$$E_h = \rho Q_E$$
$$E_v = 0.2 S_{DS} D$$

from V (application of horizontal forces simultaneously in 2 directions).
dead load at right angles.
(some exceptions)

→ So basic combinations for strength design

$$5) (1.2 + 0.2 S_{DS}) D + \rho Q_E + L + 0.2 S$$

$$6) (0.9 \cancel{+} 0.2 S_{DS}) D + \rho Q_E + 1.6 H$$

Lateral earth pressure

→ Where specifically required, conditions requiring overstrength factor →

For Load combination

$$5) E_m = E_{mh} + E_v$$

$$6) E_m = E_{mh} - E_v$$

$$E_{mh} = \Omega_0 Q_E$$

need not exceed the max force that can develop in the element as determined by a rational, plastic mechanism analysis or NL analysis.

so

$$(1.2 + 0.2 S_{DS}) D + \Omega_0 Q_E + L + 0.2 S$$

$$(1.2 - 0.2 S_{DS}) D + \Omega_0 Q_E + 1.6 H$$

→ Direction of loading :-

SDC B → permitted to be applied independently in each of the two orthogonal directions.
(Interaction effects neglected).

SDC C → Minimum as SDC B.
 If irregularity type 5 - Orthogonal combination procedure
 SDC D,E,F → Minimum as SDC + Simultaneous application of orthogonal ground motions.

→ Table 12.6-1 for "Permitted Analytical Procedures"

Structural Characteristics	ELF	RSA	RHA
1	P	P	P
2	:	:	:
3	:	:	:
4	NP	P	P

→ Modeling Criteria

- for determining seismic loads → fixed base permitted

- Effective seismic weight = D

+ 0.25 L
+ Partitions
+ Operating
+ Snow
+ Landscaping

- Spatial distribution of mass and stiffness

- Concrete → cracked sections

- Steel frame → the contribution of panel zone deformations to overall story drift

shall be included.

- For 3D, a minimum of 3 dynamic DOF (_{2 Trans}^{1 rot}) shall be included at each level.

Equivalent lateral force Procedure :-

(1)

$$V = C_s W \quad \text{defined earlier}$$

$$C_s = \frac{S_{DS}}{(R/I_e)} \neq \frac{S_{D1}}{T(R/I_e)} \quad \text{for } T \leq T_L$$

$$\frac{S_{D1}T_L}{T^2(R/I_e)} \quad \text{for } T > T_L$$

$$\cancel{\frac{0.044 S_{DS} I_e}{}} \geq 0.01$$

$$\text{If } S_1 \geq 0.6g$$

$$C_s \neq \frac{0.5 S_1}{(R/I_e)}$$

T_L = Long-period
transition
period.

If < 5 stories, $T < 0.5 \text{ sec} \rightarrow \text{Take } S_s = 1.5 \text{ to determine } C_s$.

(2)

$$(Approximate) \quad T_a = C_t h_n^x \quad h_n = \text{structural height}$$

$C_t, x \rightarrow \text{Table}$

$T \neq C_o T_a \rightarrow \text{Table}$

from analysis.

for < 12 stories, $T_a = 0.1 \times N$ no. of stories.

frames

$$\text{for masonry, concrete SW, } T_a = \frac{0.0019}{\sqrt{C_w}} \cdot h_n$$

$$\text{where, } C_w = \frac{100}{A_B} \sum_{i=1}^x \left(\frac{h_n}{h_i} \right)^2 \frac{A_i}{\left[1 + 0.83 \left(\frac{h_i}{D_i} \right)^2 \right]}$$

(check equation 12.8-10)

(3)

$$F_x = C_{vx} V$$

$$C_{vx} = \frac{w_x h_x^k}{\sum_{i=1}^n w_i h_i^k} = \text{vertical distribution factor.}$$

(4)

$$V_x = \sum F_{xi} = (\text{horizontal story Shear in any story})$$

→ Inherent torsional moment M_t (for diaphragms which are not flexible) resulting from eccentricity between locations of the center of mass, and center of rigidity.
 For flexible → distribution of forces shall account to the vertical elements.

$$\rightarrow M_t + M_{ta}$$

(Non-flexible diaphragms)

accidental
 (assumed disp of the centre of mass by a distance 5% of the structural dimension \perp to applied forces)

For type 1a or 1b torsional irregularity,
 M_{ta} should be amplified by A_x .

$$A_x = \left(\frac{S_{max}}{1.2 S_{avg}} \right)^2$$

(5) $S_x = \text{deflection at } x \text{ level} = \frac{C_d \delta_{xe}}{I_e}$ from elastic analysis under strength-level design EQ forces.

$$IDR_x = \frac{(S_{xe} - S_{(x-1)e}) C_d}{I_e}$$

Table 12.12-1 → Allowable IDR

(6) Stability Coefficient $\theta = \frac{P_x \Delta I_e}{V_x h s_x C_d}$

If $\theta < 0.1 \rightarrow$ NO need to include $P\Delta$.

$$\theta_{max} = \frac{0.5}{8 C_d} \leq 0.25$$

If $\theta_{max} < \theta < 0.1 \rightarrow$ rational analysis

Shear D/C ratio for story b/w n, n-1 levels.

OR
 \times displacements and member forces

If $\theta > \theta_{max}$
 structure unstable \rightarrow redesign by $\frac{1}{1-\theta}$

Modal RSA

→ Significant no. of modes = combined modal mass participation
 $\geq 90\%$ of actual mass in each of orthogonal directions.

→ Force-related design Parameters (Story drifts, support forces, individual member forces) for each mode = $\frac{\text{Elastic}}{R/I_e}$

$$\text{Displacement and drifts} = \left(\frac{\text{Elastic}}{R/I_e} \right) \times C_d$$

→ SRSS or CQC or CQC-4 (modified by ASCE 41)

→ If calculated $T > C_{UTa}$, Use C_{UTa} .

→ If $V_{RSA} < 0.85 V_{ELF}$, multiply forces

$$\text{with } 0.85 \frac{V_{RSA ELF}}{V_{ELF} RSA}$$

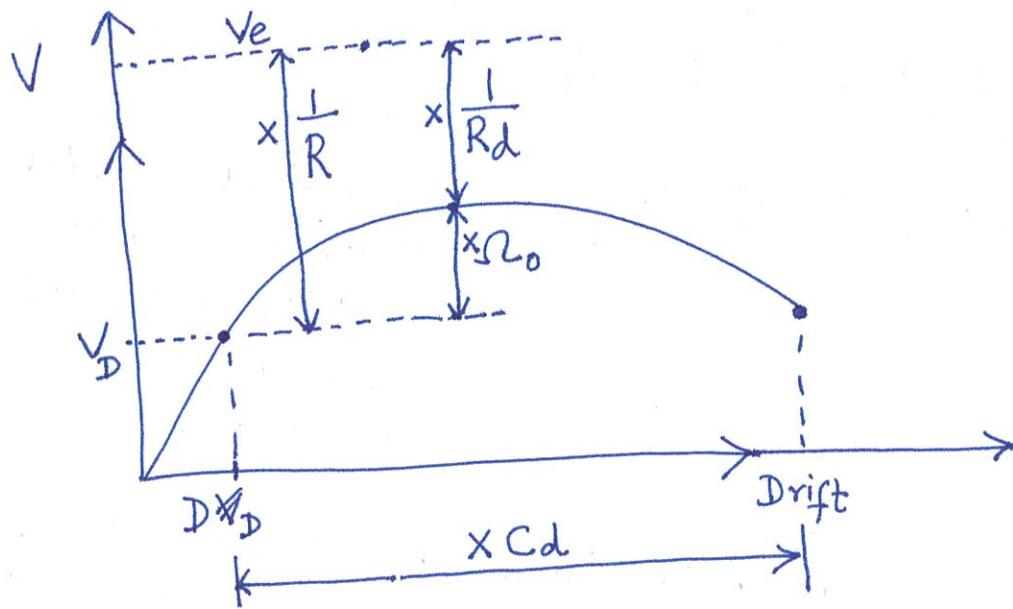
→ If $V_{RSA} < 0.85 C_s W$, multiply drifts

$$\text{with } 0.85 \frac{C_s W}{V_{RSA}}$$

→ The design story drift (Δ) $< \Delta_a$

If Moment frames + SDC D, E, F $\rightarrow \Delta < \frac{\Delta_a}{P}$

(Δ_a from Table 12.12-1)



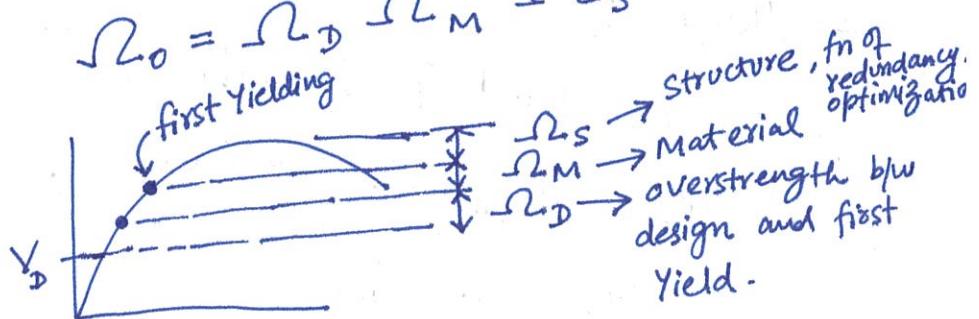
$$V_D = \frac{V_e}{R}$$

for design of force-controlled members $= V_{Des} = \frac{V_e}{R} \times \Omega_0$

$$\text{Disp and drifts} = [D_{\text{from } \frac{V_e}{R}}] \times C_d$$

NEHRP 2001 :-

$$\Omega_0 = \Omega_D \Omega_M \Omega_S$$



Seismic Load effects and Combinations

All members should be designed for these effects.

Seismic Load Effect (E) (for load Combinations)

$$E = E_h \pm E_v$$

$$= 0.2 S_{DS} D$$

↓ effect of dead load

$= \rho Q_E$

redundancy factor (1 or 1.3) effects from $\frac{V_e}{R}$

So combinations for strength design,

$$(1.2 + 0.2 S_{DS}) D + \rho Q_E + L + 0.2 S$$

$$(0.9 - 0.2 S_{DS}) D + \rho Q_E + 1.6 H$$

Seismic load Effect Including S_{20}

$$E_m = E_{nh} \pm E_v$$

↓ effect of seismic forces including S_2

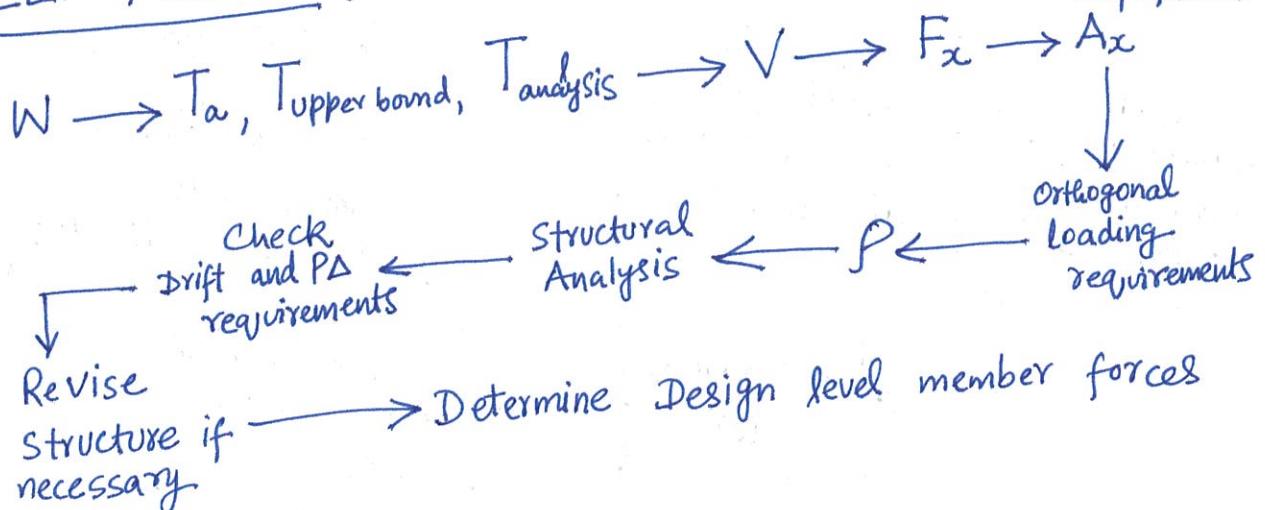
$$= S_{20} Q_E$$

↓ effect from $\frac{V_e}{R}$

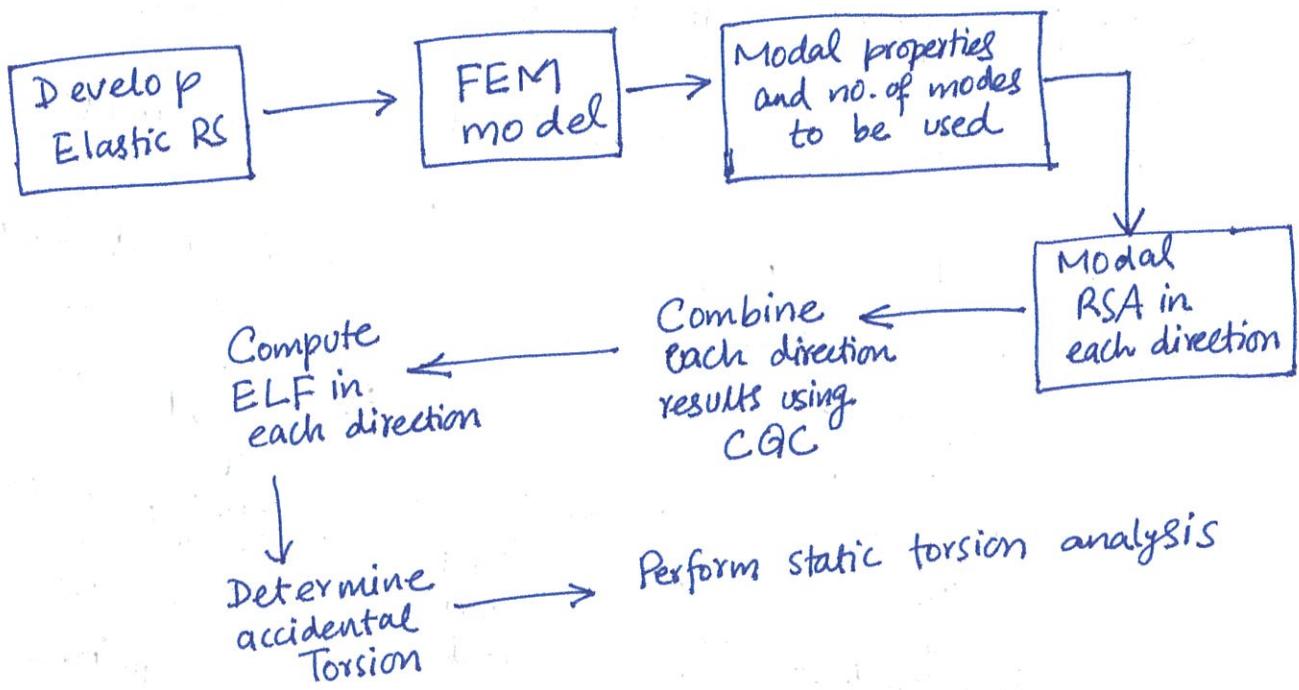
$$(1.2 + 0.2 S_{DS}) D + S_{20} Q_E + L + 0.2 S$$

$$(0.9 - 0.2 S_{DS}) D + S_{20} Q_E + 1.6 H$$

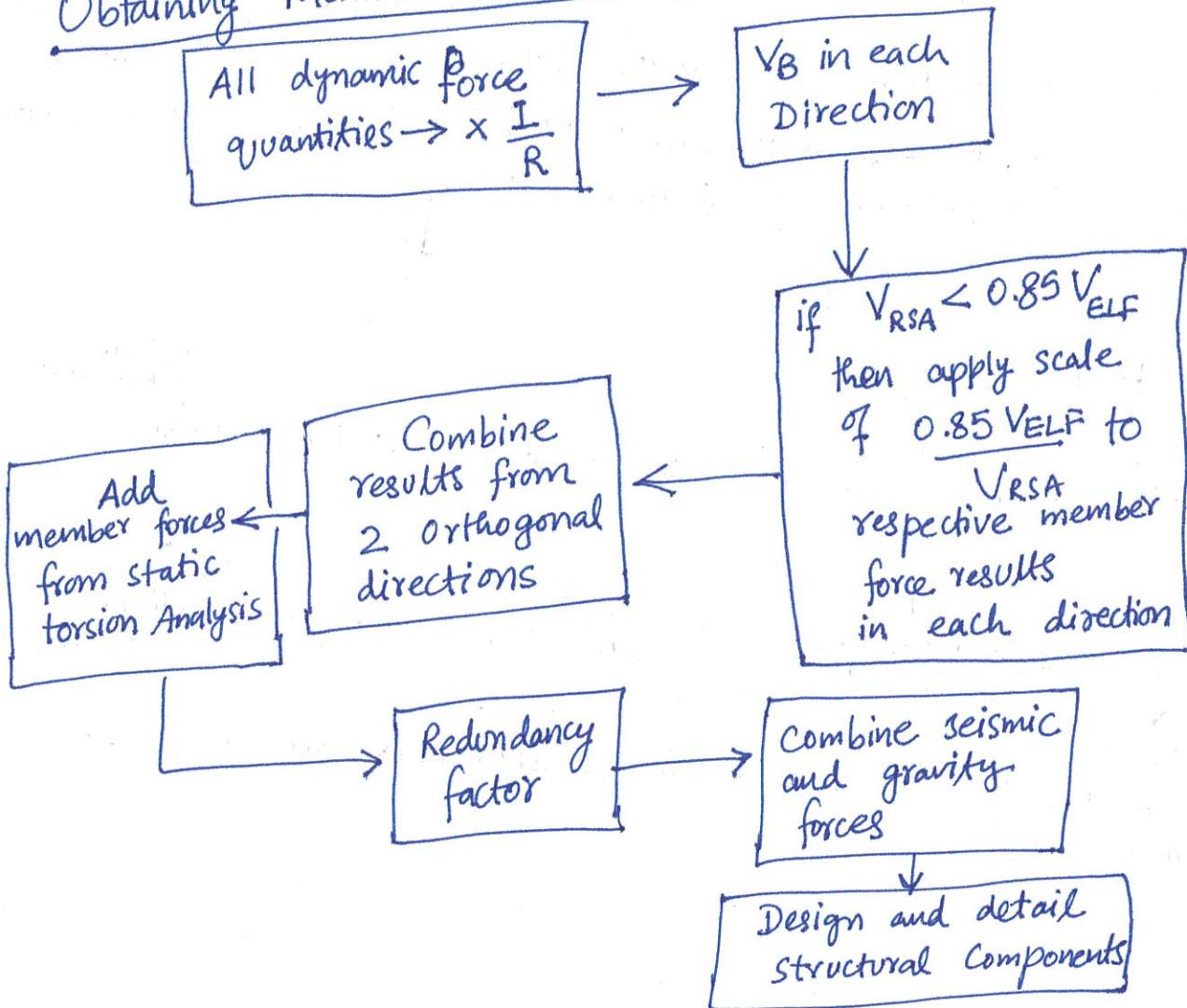
ELF Procedure :-



Modal RSA



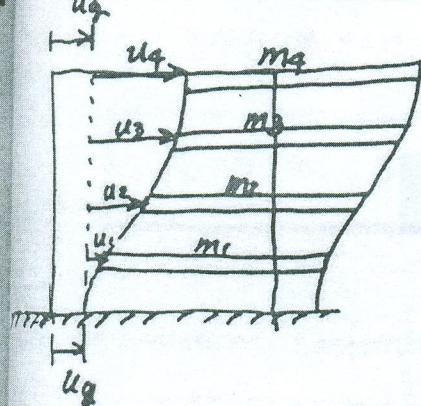
Obtaining Member Design Forces :-



RESPONSE OF MDF SYSTEMS UNDER EGs

DYNAMIC | MULTI-STORY BUILDING SUBJECTED TO EQ.

11



$$U^t = \begin{Bmatrix} u_g + u_4 \\ u_g + u_3 \\ u_g + u_2 \\ u_g + u_1 \end{Bmatrix} = \begin{Bmatrix} u_g \\ u_g \\ u_g \\ u_g \end{Bmatrix} + \begin{Bmatrix} u_4 \\ u_3 \\ u_2 \\ u_1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & \\ 1 & \\ 1 & \\ 1 & \end{Bmatrix} u_g + U$$

$\nwarrow u_g(t)$

$$U^t = \{\Pi\} u_g(t) + U(t)$$

$$\therefore \ddot{U}^t = \{\Pi\} \ddot{u}_g(t) + \ddot{U}(t)$$

Solutions of Motion:

$$M \ddot{U}^t + C \dot{U}^t + K U^t = \Phi$$

$$\omega_n^2 M \Phi = K \Phi$$

$$M \ddot{U} + C \dot{U} + K U = -M \{\Pi\} \cdot \ddot{u}_g(t)$$

transform to the principal (modal) coordinates:

$$U = \sum_{i=1}^4 \Phi_i \eta_i(t) = \Psi \eta \quad \text{where } \eta = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{Bmatrix}$$

$$\Psi^T M \Psi \ddot{\eta} + \Psi^T C \Psi \dot{\eta} + \Psi^T K \Psi \eta = -\Psi^T M \{\Pi\} \cdot \ddot{u}_g(t)$$

$$\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \end{bmatrix} \ddot{\eta} + \begin{bmatrix} \omega_1 & & & \\ 2\zeta_1 \mu_1 \omega_1 & \ddots & & \\ & \ddots & \ddots & \\ & & & \omega_4 \end{bmatrix} \dot{\eta} + \begin{bmatrix} \mu_1 \omega_1^2 \Phi_1 \\ \mu_2 \omega_2^2 \Phi_2 \\ \mu_3 \omega_3^2 \Phi_3 \\ \mu_4 \omega_4^2 \Phi_4 \end{bmatrix} \eta = -\underbrace{\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}}_{\{\Pi\}} \ddot{u}_g(t)$$

Mode i^{th} , the equation is:

$$\mu_i \ddot{\eta}_i + 2\zeta_i \mu_i \omega_i \dot{\eta}_i + \mu_i \omega_i^2 \eta_i = -L_i \ddot{u}_g(t)$$

where $L_i = \Phi_i^T \cdot \{\Pi\} = +\{\Phi_{i4} \Phi_{i3} \Phi_{i2} \Phi_{i1}\} \cdot \begin{bmatrix} M_4 & M_3 & 0 \\ 0 & M_2 & M_1 \\ \Phi_{i4} & \Phi_{i3} & \Phi_{i2} & \Phi_{i1} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$

$$L_i = \sum_{j=1}^4 M_j \Phi_{ij}$$

$M_i = \Phi_i^T M \Phi_i = \{\Phi_{i4} \Phi_{i3} \Phi_{i2} \Phi_{i1}\} \cdot \begin{bmatrix} M_4 & M_3 & 0 \\ 0 & M_2 & M_1 \\ \Phi_{i4} & \Phi_{i3} & \Phi_{i2} & \Phi_{i1} \end{bmatrix} \cdot \begin{Bmatrix} \Phi_{i4} \\ \Phi_{i3} \\ \Phi_{i2} \\ \Phi_{i1} \end{Bmatrix}$

$$M_i = \sum_{j=1}^4 M_j \Phi_{ij}^2$$

Eigen-Equation for Mode i^{th} is:

$$|K\Phi_i = \omega_i^2 M \Phi_i|$$

The Maximum Deformation in this Mode:

$$U_{\max} = \Phi_i n_{i\max}$$

Determined from
Elastic Response Spectrum

$$n_{i\max} = z_0$$

The corresponding equivalent static force:

$$f_{s0} = |K U_{\max}| = |K \Phi_i \cdot n_{i\max}|$$

$$= (n_{i\max}) \cdot |K \Phi_i|$$

$$= (n_{i\max}) \cdot \omega_i^2 M \Phi_i$$

$$= (n_{i\max}) \cdot \omega_i^2 \begin{bmatrix} M_4 & & & \Phi \\ & M_3 & & \\ & & M_2 & \\ \Phi & & & M_1 \end{bmatrix} \begin{Bmatrix} \Phi_{i4} \\ \Phi_{i3} \\ \Phi_{i2} \\ \Phi_{i1} \end{Bmatrix}$$

$$f_{s0} = (n_{i\max}) \cdot \omega_i^2 \begin{bmatrix} M_4 \Phi_{i4} \\ M_3 \Phi_{i3} \\ M_2 \Phi_{i2} \\ M_1 \Phi_{i1} \end{bmatrix} = \text{forces corresponding to } i^{\text{th}} \text{ mode.}$$

